Optimal certification policy, entry, and investment in the presence of public signals

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We explore the optimal disclosure policy of a certification intermediary where (i) the seller decides on entry and investment in product quality, and (ii) the buyers observe an additional public signal on quality. The optimal policy maximizes rent extraction from the seller by trading off incentives for entry and investment. We identify conditions under which full, partial or no disclosure can be optimal. The intermediary's report becomes noisier as the public signal gets more precise, but if the public signal is sufficiently precise, the intermediary resorts to full disclosure. However, the social welfare may reduce when the public signal becomes more informative.

1. Introduction

■   Certification intermediaries are a common feature in many markets where the consumers may not be able to readily assess the quality of the sellers' products. For example, credit rating agencies certify financial instruments, auditors certify the financial standings of organizations, numerous professional groups certify the qualifications and skills of their members, and a large numbers of agencies and laboratories offer certification service for product safety. It is also interesting to note that in such markets, the certification intermediaries are often not the only source of information as the consumer may have access to some information that is publicly available. For example, in the United States, the investors in a publicly traded company not only consider the firm's credit rating but also its filings with the Securities and Exchange Commission as they are informative about the firm's overall financial strengths. Similarly, even when a firm gets its product certified by an intermediary, other non-profit independent agencies such as *Consumer Reports* can also publicly rate its product and serve as an additional source information.

There is a vast literature on certification intermediaries that studies how their presence might improve the trade efficiencies by alleviating the adverse selection problem in the marketplace through the provision of relevant information (Lizzeri, 1999). The reduction in information asymmetry between the trading parties can also enhance the sellers' incentives to invest in quality and improves allocational efficiency (Albano and Lizzeri, 2001), but the presence of an intermediary also has redistributive effects. By charging a fee for its certification service, the intermediary can extract the surplus (or a part of it) that the seller expects to earn in the product market.

Such redistributive effects, in turn, have efficiency implications as well. Clearly, the seller's investment level, though enhanced in the presence of an intermediary, stays inefficiently low as the intermediary captures parts of the resulting gains, but the presence of an intermediary can also distort the seller's entry incentives, as the intermediary can extract more surplus than what it helps to create. A seller typically incurs a fixed cost to set up operations when breaking into a market, and such cost becomes sunk after entry. Thus, the intermediary may stymie entry if the seller's share of the trading gains falls short of its entry cost.

The extant literature on certification intermediaries has not fully explored the interplay between such redistributive and the efficiency effects as the extent of entry is usually assumed to be exogenously given. In this article, we consider a setting where the level of entry by the sellers is endogenous. In such an environment, the presence of the intermediary gives rise to a novel trade-off: it enhances a seller's incentives to invest in quality following entry but mutes his incentive to enter in the first place. The goal of this article is to explore the intermediary's optimal certification policy in the face of this trade-off. We also analyze how the informativeness of the intermediary's signal interacts with the precision of the public signal on quality and draw out its welfare implications.

We consider a model of certification intermediary with the following key features. The intermediary commits to a policy that specifies a certification fee and a disclosure policy (we will elaborate on this shortly). Upon observing the certification policy, the seller decides whether to enter by incurring a fixed cost, and following entry, whether to invest to improve its product quality. The investment increases the likelihood of producing a high quality product, and the cost of investment depends on the seller's private type. After the quality is realized, the seller decides whether to use the intermediary's service. The buyers cannot directly observe the quality but can obtain relevant information from two sources: a public signal whose precision is exogenously fixed, and the intermediary's signal (i.e., whether the seller has certified his product, and if so, what signal the intermediary has released). The buyers offer a bid for the product that is driven by their belief about the product quality, given the available information.

As in Lizzeri (1999), we define a disclosure policy as a probability distribution over a set of signals conditional on the quality of the seller's product. Such a specification accommodates the extreme cases of full and no disclosure as well as the more generic case of partial disclosure where the intermediary reveals a garbled information about the underlying product quality. Also, in order to highlight the interplay between the entry and investment incentives in the most transparent way, we restrict parameters such that entry is always efficient but investment may be inefficient in the absence of the intermediary.

We derive three sets of results. First, we characterize a "full-disclosure" benchmark where the intermediary is required to reveal the quality without any noise should the seller opt to use its service. In particular, we argue that the optimal certification fee distorts entry if and only if the entry cost is sufficiently large (Proposition 1).

The intuition is as follows. With mandated full disclosure, the intermediary's problem is similar to that of a "standard" monopolist—by charging a higher certification fee, the intermediary can extract more rents from the seller but also reduces the likelihood that the seller would choose to certify his product. Note that with full disclosure, only a seller with high quality product may opt to certify. Thus, in order to ensure that the intermediary faces a robust demand for its service, it must protect both the entry and the investment incentives of the seller. In other words,

the intermediary chooses the certification price so as to trade off the gains from rent extraction with the losses from both the entry and the investment distortions.

When the entry cost is low, the optimal certification fee does not affect the seller's entry incentives, but as the entry cost increases, the incentives for entry may diminish and the intermediary must reduce its certification fee in order to induce full entry. However, when the entry cost is sufficiently large, accommodating entry for all types of the seller is too costly. In response, the intermediary chooses a price such that entry is viable only if the seller also invests in quality (following entry), and all seller types with relatively high investment cost are foreclosed from the market.

Our second key finding offers a characterization of the intermediary's optimal disclosure policy. For expositional clarity, we state the key features of the optimal policy in a series of propositions (Propositions 2, 4, and 5), and present the analysis in two steps. We first consider the case where the public signal is completely uninformative, and then study the implications of an informative public signal.

We show that in the absence of any public signal, the optimal disclosure policy offers full disclosure only if the entry cost is sufficiently small but calls for partial disclosure otherwise. In the optimal partial disclosure policy, the low quality seller, with some probability, receives the same certification that the high quality seller gets. Moreover, entry is always efficient, and under partial disclosure, the seller always uses the intermediary irrespective of his product quality.

To see the argument, recall that under full disclosure, the optimal certification fee trades off the gains from rent extraction with the losses from diminished incentives for both entry and investment, and when it is optimal to induce full entry, the intermediary significantly lowers its certification fee in order to incentivize entry of all types. As a result, it ends up leaving a large amount of rent with the high-quality seller. In contrast, a noisy disclosure can allow the intermediary to extract more rents from the seller where the resulting damage to entry incentives is partly restored through a garbling of information. By partly pooling the low and high quality seller, the intermediary can increase its payoff by ensuring efficient entry and inducing the seller to use its service irrespective of the realized quality level. This is due to the fact that under partial disclosure, even a low quality seller expects to be pooled with the high quality and fetch a higher bid from the buyers, whereas the buyers may believe the seller to be of low quality if he does not use the intermediary.

Next, we allow for an informative public signal. In this case, the optimal policy becomes considerably more nuanced. In particular, as the precision of the public signal increases, the intermediary's report gets increasingly more noisy, and eventually becomes absolutely uninformative. Such a partial or no disclosure policy remains optimal only for a moderate range of entry costs (full disclosure being optimal otherwise), and this range decreases with the public signal's precision. Once the public signal becomes sufficiently precise, the intermediary resorts to full disclosure irrespective of the cost of entry. In other words, the public signal and the intermediary's report interact as substitutes as long as the public signal is not too precise, but otherwise, they become complements.

The intermediary offers partial disclosure so as to manipulate the spread between the buyers' bids that a high- and a low-quality seller expect to receive in the product market, but as the intermediary's signal is (weakly) informative, its ability to influence the buyers' belief, and hence, their bids, is constrained by the precision of the public signal. Until this constraint is binding, the intermediary adds more noise to its signal as the public signal becomes more precise and implements the desired spread in the bids, but once this constraint becomes binding, it is optimal for the intermediary not to disclose any further information.

However, when the public signal is relatively precise, the low-quality seller anticipates a low bid for the buyers (irrespective of how noisy the intermediary's report is). So, if the intermediary were to induce all types of sellers to enter and use its service, even with partial disclosure, it must offer a significant reduction in the certification fee. Instead, it may be more profitable for

the intermediary to switch back to full disclosure and charge a higher fee in order to extract rents from the high-quality sellers only.

Our third key result analyzes how the social welfare under the intermediary's optimal policy varies with the precision of the public signal (Proposition 6). We show that, contrary to the common intuition, a more precise public signal on quality can lead to a decrease in the social welfare. These findings stem from the fact that an increase in the precision of the public signal can lead to a regime change in the intermediary's disclosure policy where it switches from partial disclosure with full entry to full disclosure policy with restricted entry (when the entry cost is sufficiently large). The resulting welfare loss due to the entry inefficiency can outweigh the gains due to stronger investment incentives, and can lead to an overall drop in the social welfare.

Related literature: There is by now a large literature on the role of certification intermediaries in markets plagued by asymmetric information (some early contributions include, among others, Biglaiser, 1993; Biglaiser and Friedman, 1994; Lizzeri, 1999; and Albano and Lizzeri, 2001; also see Dranove and Jin, 2010, for a survey). As briefly discussed earlier, Lizzeri analyzes the role of certification intermediaries in a model of adverse selection, and shows that the optimal choice for the intermediaries often entails no disclosure or partial closure in the form of minimum quality certification. Although Lizzeri assumes that the seller's product quality is exogenously fixed, Albano and Lizzeri (2001) extends Lizzeri's model to endogenize the quality. They analyze the issue of the optimal degree of information revelation and show that the presence of intermediary enhances efficiency by increasing the sellers' incentives to provide high quality (though the full information first-best allocation remains infeasible).[1] Dubey and Geanakoplos (2010) also consider the optimal grading scheme in inducing students' efforts and show that coarse grading (i.e., partial disclosure) often motivates the students to work harder when the students care about their relative rank in the class,[2] but in contrast to our setup, these models abstract away from the question of the seller's entry incentives and the interplay between the public signal on quality and the intermediaries' disclosure policy (i.e., they assume that the seller is already in the market and the intermediary is the only source of information for the buyers).

A recent article by Harbaugh and Rasmusen (2018) explores how certification intermediaries may influence the sellers' entry decision. As in our article, they show that coarse grading—that is, partial disclosure—is used to induce more participation by the sellers. However, they consider a pure adverse selection model without investment. More importantly, they assume a non-profit certifier whose objective is to enhance the amount of information available to the buyers when the seller incurs an exogenous certification cost. Our model, in contrast, assumes a profit-maximizing certifier with an endogenous certification price, and the cost of garbling information comes from reduced investment incentives rather than from the loss of information per se.[3]

Our article is also related to the literature on the optimal design of information structure (Ostrovsky and Schwarz, 2010; Kamenica and Gentzkow, 2011; Rayo and Segal, 2010).[4] Kamenica and Gentzkow analyze the general Bayesian persuasion problem in which a sender chooses the optimal information structure for a signal to be revealed to a receiver, and derive general conditions under which the sender may benefit by controlling the informational environment. In the specific context of monopoly pricing, Roesler and Szentes (2017) consider an information design problem in which a single buyer can design her own information about her willingness-to-pay before she faces a monopolist seller. As in the Bayesian persuasion literature, we also assume that the intermediary can precommit to a particular disclosure policy. However, in our setting

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[1] In a related article, Belleflamme and Peitz (2014) assume that consumers observe the investment (but not the realization of the product quality) and show that the firm overinvest in quality compared to the full information benchmark.

[2] Similar issues are also discussed by Costrell (1994) and Boleslavsky and Cotton (2015).

[3] Hui et al. (2018) empirically explore a related issue. Using a change in the certification policy at eBay as a natural experiment, they study how the policy may affect the entrants' types and the sellers' behavior. However, in their setting, the policy is set exogenously, and the certification is offered by eBay for free.

[4] See Bergemann and Morris (2019) for an excellent survey that presents a unified approach to the recent information design literature. For an analysis of a Bayesian persuasion game with voluntary participation, see Rosar (2017).

the intermediary's problem cannot be modeled as a standard sender's problem because the distribution of the product quality (i.e., the underlying "state of the world" that the sender reveals information on) is endogenous to the sender's disclosure policy.[5]

Two articles in the information design literature are particularly relevant to our work: Kolotilin et al. (2017) and Guo and Shimaya (2019). Both of these articles analyze the information design problem with a privately informed receiver. As in our analysis, they consider a situation in which the receiver has multiple sources of information. However, their focus is on the case where the receiver's additional information is private and it leads to a monopoly screening problem. Kolotilin et al. (2017), for instance, consider a setting in which the receiver has private information about his preference type, and address the question of whether the sender can benefit from designing a complex persuasion mechanism that conditions information disclosure on the receiver's report about his type. They demonstrate the equivalence of implementation by persuasion mechanisms and by simple experiments that disclose information independent of the receiver's type. In Guo and Shimaya's (2019) analysis, the receiver has private information about the quality of the project the sender persuades him to accept. They show that the optimal screening mechanism is characterized by a nested interval structure. These articles, however, do not assume any monetary transfer for information disclosure, as is typical in the Bayesian persuasion literature.

In contrast, in our model the information is sold at an endogenously determined price and the price of information is an important consideration in the choice of disclosure policy. Like our article, Bergemann et al. (2018) consider a monopolistic seller of information about a state variable that is relevant to the buyer's decision. In their model, the buyer has private knowledge about his decision problem at the time of contracting. Instead, we consider a public signal as another source of information that arises after contracting, thereby abstracting from the issue of screening.

Finally, it is worth noting that our analysis on the interplay between the informativeness of the intermediary and the precision of the public signal is related to the literature on the adverse consequences of transparency (if we interpret more precise public signal as more transparency in an agency relationship). Prat (2005), for instance, shows that an agent with career concerns may ignore his signal, to the detriment of the principal, and behave as a conformist when his action is observed (i.e., transparent). Levy (2007) considers the effect of transparency on committee decisions and identifies circumstances under which a secretive committee that uses a particular voting rule makes better decisions on average. Their analyses, however, are in completely different contexts and rely on career-based reputation effects.

The remainder of the article is organized as follows: In Section 2, we present our model. A benchmark case of full disclosure is analyzed in Section 3. For expositional clarity, we present the analysis of the optimal certification policy in two steps. First, in Section 4, we characterize the optimal disclosure policy when the public signal is completely uninformative. Next, in Section 5, we analyze the general case with an informative public signal. The welfare implications of the intermediary's optimal policy is explored in Section 6. Section 7 presents a conclusion. All proofs are given in the Appendix.

2. Model

■   *Players*. We consider an environment with three types of players: a seller, a certification intermediary, and a set of identical buyers.

Actions and payoffs. The seller decides whether to enter a market to sell his product to a set of identical buyers. The buyers' valuation of the seller's product is based on its quality. The product

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[5] Boleslavsky and Kim (2018) studies optimal information design when an agent's private effort determines the distribution of an unobservable state. In contrast to their setup, we also have an element of adverse selection with heterogeneous agent types and additional constraint on the entry.

quality could be either high or low and generates a value $v \in \{0, 1\}$ for the buyer, where $v = 0$ if the quality is low and $v = 1$ if it is high.

If the seller decides to enter the market, he incurs an entry cost of k. The entry cost may be interpreted as a sunk cost that the seller pays to set up his operations in the marketplace regardless of the quality of his product.[6] Upon entry, he can undertake an investment in quality in order to increase the likelihood of producing a high quality product. Let $I \in \{0, 1\}$ denote the seller's investment decision where $I = 1$, if he invests in quality and $I = 0$ otherwise. We have

$$\Pr(v = 1 \mid I = 1) = \alpha > \frac{1}{2} = \Pr(v = 1 \mid I = 0). \tag{1}$$

The cost of this investment is determined by the seller's type $\theta \in [0, 1]$, which is assumed to be uniformly distributed on $[0, 1]$. For a seller of type θ, the cost of investment is c/θ for $\theta \neq 0$ and $C(> 1)$ for $\theta = 0$. The seller's type is his private information and known to him before he makes his entry decision.[7,8]

The product quality is also privately observed by the seller, leading to an information asymmetry in the product market, but the buyers can obtain information on quality through two channels.

First, the seller can hire a certification intermediary who can verify the quality and disclose additional information for a fee. The intermediary is assumed to be a monopolist in the market for certification services, and it verifies the quality without incurring any cost. At the beginning of the game, the intermediary commits to a certification price p and a disclosure policy $\mathcal{D}$ that specifies what it may disclose to the buyers, given the underlying quality of the product. However, the intermediary may not fully reveal the quality of the product and can potentially garble its report. In order to allow for such a noisy disclosure, we define a disclosure policy as a mapping $\mathcal{D} : \{0, 1\} \to \Delta X$ where X is a pre-specified signal space (and hence, a part of the policy). That is, the disclosure policy sends a signal $x \in X$ that is drawn according to a given probability distribution, conditional on the true quality of the product.

Second, in addition to the intermediary's signal, the buyers observe a public signal $z \in \{0, 1\}$ that provides noisy information about the product quality, where

$$\Pr(z = j \mid v = j) = \pi \in [1/2, 1), \text{ for } j = 0, 1.$$

The parameter π represents the precision of the public signal, when $\pi = \frac{1}{2}$ the pubic signal is completely uninformative.

The seller makes his entry decision after observing the intermediary's offer $(p, \mathcal{D})$. Moreover, the seller decides on whether to hire the intermediary after the quality realization but before the public signal is realized. The intermediary, if hired, also reveals its report before the arrival of the public signal.[9] Observing the available signals on the quality, the buyers simultaneously bid for the product, and the product is sold at the highest bid. All players are assumed to be risk neutral. This implies that the product is sold at the expected value of the product given the information available to the buyers.

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[6] We maintain the same interpretation of the entry cost as used in the canonical models of a firm's entry decision; see, for example, Spence (1976) and Mankiw and Whinston (1990).

[7] For simplicity, we normalize the seller's probability of producing a high quality product without any investment to $\frac{1}{2}$. Also, if $\Pr(v = 1 \mid I = 0) < \frac{1}{2} < \alpha$, we may not have a unique equilibrium even in the absence of the intermediary, and the characterization of the optimal policy becomes analytically intractable.

[8] As no new information is revealed in the time between the entry and investment decisions, one can reinterpret these two decisions as a choice between two modes of entry—entry with a "standard" technology at cost k, and entry with an "advanced" technology at cost $k + c/\theta$, where the latter mode is more likely to yield a high quality product.

[9] As z is realized after the intermediary reveals its signal x, one can also interpret z as a signal that is *privately* but commonly observed by the buyers.

Timing. The following timeline summarizes the game:

- *Stage 1*: The intermediary commits to his certification policy $(p, \mathcal{D})$.
- *Stage 2*: The seller observes his type θ and the intermediary's policy, and decides whether to enter by incurring a cost of k. (If there is no entry, the game ends.)
- *Stage 3*: If the seller enters, he decides whether to invest on quality at cost c/θ, and observing his product quality v, decides on whether to hire the certification intermediary.
- *Stage 4*: The intermediary, if hired, reveals its signal x on the product quality.
- *Stage 5*: The public signal z on quality is revealed. Observing x (if available) and z, the buyers bid for the product, and the product is sold at the highest bid.

Strategies and equilibrium concept. The strategies of the players are as follows: the intermediary's strategy is to choose a certification policy $(p, \mathcal{D})$. The seller's strategy has three components: (i) entry and (ii) investment decisions given his type and the intermediary's policy, and (iii) decision on hiring the intermediary given his product quality and the intermediary's policy. Finally, the buyers' strategy is to choose a bid given the available information (i.e., the public signal, the intermediary's certification policy, whether the intermediary was hired or not, and if hired, the intermediary's report). We use (*pure strategy*) perfect Bayesian Equilibrium (PBE) as the solution concept. The optimal disclosure policy is defined as the $(p, \mathcal{D})$ pair that induces the highest feasible equilibrium payoff for the intermediary.

   We maintain the following parametric restrictions to streamline our analysis. Denote $\Delta := \alpha - \frac{1}{2}$.

Assumption 1. (*i*) $\sqrt{c\alpha} < \Delta$; (*ii*) $\frac{c}{2\Delta} < k < \frac{1}{2}$.

   Assumption 1 implies that when the seller's product quality is publicly observable, that is, in the first-best scenario, the seller should always enter irrespective of his type (as $k < \frac{1}{2}$), and all types of the seller above the threshold $\theta^{FB} = \frac{c}{\Delta}$ should invest. We impose a stronger restrictions on the parameters than simply requiring $c < \Delta$ and $k < \frac{1}{2}$ so as to simplify our analysis. The assumption above allows us to focus on the part of the parameter space where the interplay of entry and investment decisions is non-trivial and gives rise to a rich set of equilibrium characteristics.

3. Full disclosure

■   We begin our analysis by exploring a simple case where the intermediary must fully disclose the product quality (i.e., the intermediary only sets the price for its service, and the seller decides whether or not to certify her product). The analysis of the full disclosure case clearly illustrates the trade-offs that the intermediary faces when choosing its certification fee, and it is instructive in understanding when and how a partial disclosure policy may be optimal.

   What is the intermediary's maximal equilibrium payoff under such a policy? The following two observations considerably simplify our analysis. First, for any given certification policy of the intermediary, the seller's entry and investment decisions follow a cutoff rule: there exist cutoffs θ_E and θ_I such that $\theta_E \leq \theta_I$, all types $\theta \geq \theta_E$ enter, and all types $\theta \geq \theta_I$ invest. This observation directly follows from the fact that as the seller's cost of investment is monotonically decreasing in his type (θ) and all types face the same entry cost (k). Second, in the best equilibrium (with full disclosure) for the intermediary only the high quality seller certifies its product, and the buyers take a "skeptical posture" à la Milgrom and Roberts (1986)—they believe the product to be of low quality if the seller does not hire the intermediary. Thus, the certification price p that maximizes the intermediary's payoff solves a "standard" monopolist's pricing problem where the market demand is $\Pr(v = 1 \mid \theta_E, \theta_I)$—the probability that a high quality seller arrives on the market (the marginal types for entry, θ_E and investment θ_I depend on the intermediary's certification price p).

The intermediary can adopt one of two possible pricing strategies: (i) Set $p \leq p_E := 1 - 2k$ in order to induce all types of the seller to enter.[10] As a result, we have $\theta_E = 0$, and $\theta_I = c/(1 - p)\Delta$, that is, the type that is indifferent between investing and not investing conditional on already being on the market. (ii) Set $p > p_E$ such that lower types stay out and all types that enter also invest in quality. Here, $\theta_I = \theta_E$, and $\theta_I = c/(\alpha(1 - p) - k)$, that is, the type that is indifferent between entering the market and investing after entry and not entering at all.

The intermediary's maximization problem, therefore, is given as:

$$\mathcal{P}_F : \begin{cases} \max\limits_{p \geq 0} \Pi(p) := p \Pr(v = 1 \mid \theta_E, \theta_I) \\ s.t.\ \theta_E = \begin{cases} 0 & \text{if } p \leq p_E \\ \theta_I & \text{otherwise} \end{cases}, \text{ and } \theta_I = \begin{cases} c/(1 - p)\Delta & \text{if } p \leq p_E \\ c/(\alpha(1 - p) - k) & \text{otherwise} \end{cases} . \end{cases}$$

Let p^* be the solution to the problem $\mathcal{P}_F$.

Proposition 1. (**Optimal pricing under full disclosure**) Under full disclosure, the optimal certification price distorts entry if and only if the entry cost is sufficiently large. In particular, there exists an entry cost cutoff k^* such that $p^* > p_E$ iff $k > k^*$.

The intermediary faces a trade-off between rent extraction from the seller and preserving the incentives of the seller to use the intermediary. By charging a high certification price the intermediary can extract more rent from the seller but it also reduces the demand for the intermediary's service. As only the high-quality seller hires the intermediary (under full disclosure), the intermediary benefits when the seller has a strong incentive to enter and invest in quality. However, the larger is the certification price, the weaker is the seller's incentive to invest as the intermediary keeps a larger share of the trade surplus. Moreover, if there is not enough surplus left for the seller to cover his entry cost, he may not enter the market in the first place. Thus, the intermediary's optimal pricing balances the gains from rent extraction and the losses from the reduced demand for its certification service.

Consider the optimal certification price p' (say) that the intermediary would have charged if the seller were already on the market. In such a setting, the optimal price trades off the gains from the rent extraction with the losses that emanates only from a weakened investment incentives of the seller. If the entry cost is sufficiently low, the optimal price p' is lower than p_E, the certification price below which entry remains efficient. Therefore, even in our setting the intermediary would charge $p^* = p'$, and all types of the seller would be on the market.

But p_E is decreasing in the entry cost k. So, for a larger k, we have $p' > p_E$, and the seller may not enter if his type (θ) is sufficiently low (i.e., cost of investment is sufficiently high). As long as the entry cost is moderately low (i.e., $k < k^*$), the intermediary is better off by lowering its price to p_E. By charging a lower certification price, the intermediary forgoes a part of the rent that it could have extracted from the seller, but such a loss is more than offset by the gains from increased likelihood of having a high-quality seller as the reduction in the certification price restores full entry and also strengthens the seller's incentives to invest.

However, when the entry cost is relatively large, accommodation of entry considerably hurts the intermediary as it would require a significant reduction in the certification price. As a result, when the entry cost crosses a threshold (k^*), the intermediary finds it optimal to restrict entry by raising his certification price such that all types that enter also have incentive to invest in quality.

The analysis of the full-disclosure benchmark suggests that the optimal certification policy may call for partial disclosure if it can allow the intermediary to extract more rents from the seller without distorting entry. Below, we explore the optimal disclosure policy, and to facilitate the exposition, we present our analysis in two steps. First, we study the optimal disclosure policy

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[10] The *ex ante* expected profit of a seller who enters (incurring a cost of k) but does not invest is $\frac{1}{2}(1 - p) - k$ (with probability $\frac{1}{2}$ his product turns out to be of high quality, in which case he hires the intermediary by paying p and receives a bit of 1 from the buyers). Note that $\frac{1}{2}(1 - p_E) - k = 0$.

when the public signal is completely uninformative. This step offers a sharp insight on how partial disclosure can soften the trade-off between rent extraction and the seller's incentives for entry and investment. Next, we allow for the informative public signal, and explore how it affects the intermediary's behavior in the face of the aforementioned trade-off.

4. Optimal disclosure policy in the absence of the public signal

■   In this section, we assume that the public signal z is completely uninformative, that is, $\pi = \frac{1}{2}$, and characterize the optimal certification policy. The analysis must contend with a technical challenge: The set of disclosure policies for the intermediary is considerably large as both the signal space X and the mapping $\{0, 1\} \rightarrow \Delta X$ are endogenous to the model. Consequently, it is seemingly difficult to solve for the optimal policy explicitly. Instead, we explore the intermediary's problem in terms of the set of (expected) bids from the buyers that can arise from various disclosure policies, and upon obtaining the solution in terms of the bids, we characterize a disclosure policy that results in (i.e., implements) those bids.

As the buyers bid competitively for the product, the equilibrium bid is simply the expected quality of the product given the disclosure policy, the set of types who enter, Θ_E, and invest, Θ_I, and the realized signals on quality, (x, z). For a given disclosure policy, let t_i be the *ex ante* expected bid (or "transfer" from the buyers) that the seller of quality i receives when he uses the intermediary (but yet to learn the realization of the signals x and z). That is,

$$t_i = \mathbb{E}_{(x,z)|v=i}\mathbb{E}_v[v \mid x, z, \Theta_E, \Theta_I], \quad i \in \{0, 1\}. \tag{2}$$

The intermediary's certification policy—certification price p and disclosure policy $\mathcal{D}$—can be represented by the triplet (p, t_0, t_1). Under a strictly partial disclosure policy, we have $0 < t_0 < t_1 < 1$, whereas under full disclosure, $t_0 = 0$ and $t_1 = 1$. As noted earlier, in any equilibrium, the set of types that enter and invest are pinned down by their respective cutoffs, θ_E and θ_I (say), so $\Theta_E = [\theta_E, 1]$ and $\Theta_I = [\theta_I, 1]$. Also, as we assume for now that the public signal is completely uninformative, its realization (z) does not affect the buyers' belief.

The following lemma further simplifies our analysis.

Lemma 1.   If a partial disclosure policy is optimal, it must entail "full market coverage": in equilibrium, (i) all types (θ) of the seller enter the market, and (ii) the seller hires the intermediary irrespective of his product quality.

This result stems from the following two observations: First, if the intermediary prefers to sell its service only to the high quality seller, it is best to maximize the difference in bids that a low and a high quality seller would receive from the buyer. Clearly, this is achieved through full disclosure as it removes the information asymmetry between the buyers and the seller. Second, in any equilibrium, if only some types of the seller enter it must be the case that those types who enter also invest,[11] but if the intermediary only intends to induce entry of those types who would invest as well, it can be argued that it is again (weakly) optimal to use full disclosure. Thus, if partial disclosure is indeed optimal, it must induce full market coverage.

An important implication of Lemma 1 is that in order to solve for the optimal disclosure policy, without loss of generality, we can look for the optimal partial disclosure with full market coverage, and compare the associated payoff of the intermediary with the its full disclosure counterpart.

We maintain the buyers' off-equilibrium belief that is most favorable to the intermediary, i.e., if the seller chooses not to use the intermediary, the buyers believe the quality to be low

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[11] As the entry cost is the same for all types, if its profitable for some type θ to enter even though it would not invest following entry, then entry must be profitable for all types of the seller.

and bid 0. Thus, for all types of the seller to enter and certify regardless of the realized product quality, the certification price p must satisfy the following two constraints:

$$\frac{1}{2}(t_0 + t_1) - k - p \geq 0, \tag{IR$_E$}$$

and

$$t_0 - p \geq 0. \tag{IR$_C$}$$

The (IR$_E$) constraint is the individual rationality condition for entry and states that it is profitable for all types (θ) of the seller to enter the market even if he decides not to invest in quality. The (IR$_C$) constraint ensures that the seller has incentives to certify even when his product quality is low. So, for any disclosure policy (t_0, t_1) with full market coverage, the intermediary's payoff is:

$$\Pi(p) = p = \min\left\{t_0, \frac{1}{2}(t_0 + t_1) - k\right\}.$$

Let $v(\theta_I)$ be the "prior" expected value of the product—that is, the probability that the seller's product is of high quality (i.e., $v = 1$) given that all types enter and all types above θ_I invest, but without any information on the signals. That is,

$$v(\theta_I) := \mathbb{E}[v \mid \Theta_E = [0, 1], \ \Theta_I = [\theta_I, 1]] = \frac{1}{2}\theta_I + \alpha(1 - \theta_I).$$

Now, any disclosure policy (t_0, t_1) must satisfy the following two conditions: First, a Bayes rationality (BR) condition that requires the expected posterior mean of the quality must be equal to its prior mean:

$$v(\theta_I)t_1 + (1 - v(\theta_I))t_0 = v(\theta_I). \tag{BR}$$

Second, if it is incentive compatible for all types $\theta \geq \theta_I$ to invest, the marginal type (if strictly below 1) must be indifferent between investing and not investing. That is, we must have $(t_1 - t_0)\Delta = c/\theta_I$, or,

$$\theta_I = \min\left\{\frac{c}{(t_1 - t_0)\Delta}, 1\right\}. \tag{IC}$$

Hence, the intermediary's optimal *partial* disclosure policy solves the following program:

$$\mathcal{P}_P: \quad \max_{t_0, \ t_1, \ \theta_I; \ t_0 \leq t_1} \quad \min\left\{\frac{1}{2}(t_0 + t_1) - k, t_0\right\} \quad s.t. \ (BR), \ and \ (IC).$$

We can now characterize the intermediary's optimal policy by comparing his payoff in $\mathcal{P}_F$ and $\mathcal{P}_P$, that is, the optimal payoff under full disclosure and partial disclosure (with full market coverage).

Proposition 2. (**Optimal policy without public signal**) The optimal certification policy ensures efficient entry but leads to inefficiently low investment. Moreover, there exists an entry cost cutoff $k' < k^*$ (where k^* is as defined in Proposition 1) such that:

(i) For $k \leq k'$, the intermediary offers full disclosure, and only the high-quality seller certifies his product.

(ii) For $k > k'$, partial disclosure is strictly optimal, and the seller certifies his product irrespective of the quality.

Proposition 3. (**Implementation**) The optimal disclosure policy can be implemented by the following signal structure: $X = \{x_0, x_1\}$ and

$$\Pr(x = x_1 \mid v) = \begin{cases} 1 & if \ v = 1 \\ \rho & if \ v = 0 \end{cases}.$$

Propositions 2 and 3 show that the optimal policy has a simple feature. Partial disclosure is used only if the entry cost is sufficiently large. In fact, partial disclosure may be called for even when the optimal certification fee under full disclosure would have induced full entry (note that $k' < k^*$). Also, the partial disclosure can be implemented by a binary signal structure where a "low" rating (x_0) surely indicates low quality but a "high" rating (x_1) is an inconclusive signal of quality. The intermediary always gives the high quality seller the "high" rating (x_1), but the low quality also gets the same high rating with some probability.

To see the intuitions for these results, recall the trade-off with certification price under full disclosure. A larger price extracts more rents from the seller (who uses the intermediary) but reduces the likelihood that the seller would use the intermediary in the first place. The reduction in the demand for the intermediary's service stems from the fact that when the intermediary extracts a larger share of the trade surplus, both the investment and the entry incentives of the seller are muted.

This trade-off could be softened with partial disclosure. By partially pooling the low quality seller with the high-quality one, the intermediary can ensure that an entrant gets a relatively high bid from the buyer even if he ends up with a low quality product. Thus, such a policy can induce all types of the seller to enter and use the intermediary irrespective of the product quality even if the intermediary charges a moderately high certification fee. (Though partial disclosure dampens investment incentives, it does not affect the intermediary's payoff as the seller uses the intermediary irrespective of his product quality.) The resulting payoff dominates its counterpart under full disclosure when the entry cost is sufficiently large. Indeed, when the entry cost is high, the intermediary's payoff under full disclosure gets dampened as the intermediary either sharply lowers its price in order to induce entry for all types of the seller (when k is large but still below k^*) or forecloses the market for the types with relatively high investment costs (if $k > k^*$).

5. Optimal disclosure policy with public signal

■ How would the optimal certification policy change if the public signal z is indeed informative, that is, what if $\pi \in (1/2, 1)$? As before, we can obtain the optimal policy by comparing the intermediary's optimal payoffs under full and partial disclosure. Clearly, the analysis of full disclosure remains unaltered (recall that under full disclosure public signal do not affect the buyers' beliefs both on and off the equilibrium path.), but as we show below, the optimal partial disclosure policy may depend on the public signal's precision.

Notice that in order to solve for the optimal partial disclosure policy, we can again limit attention to policies with full market coverage (Lemma 1 continues to hold as its argument does not depend on the precision of the public signal). As before, the optimal policy abides by the (BR) and incentive compatibility (IC) constraints, but in addition to these two constraints, the intermediary's program must also account for the fact that the public signal puts a bound on how much the intermediary can influence the buyers' posterior beliefs.

As the intermediary's report must be weakly informative (i.e., its signal cannot take away the information already contained in the public signal), we have the following bounds on the expected bids t_i:

$$t_1 \geq \underline{t}_1(\theta_I) := \mathbb{E}_{z|v=1}\mathbb{E}_v[v \mid z, \ \Theta_E = [0, 1], \ \Theta_I = [\theta_I, 1]], \qquad (L_1)$$

and

$$t_0 \leq \bar{t}_0(\theta_I) := \mathbb{E}_{z|v=0}\mathbb{E}_v[v \mid z, \ \Theta_E = [0, 1], \ \Theta_I = [\theta_I, 1]]. \qquad (U_0)$$

That is, the *ex ante* expected bid received by the high-quality seller when he uses the intermediary (t_1) cannot be lower than the bid he expects to receive when the buyers rely on the public signal only ($\underline{t}_1(\theta_I)$). Similarly, for the case of the low-quality seller, we have an upper bound on t_0.

Hence, the intermediary's optimal partial disclosure policy solves the following program:

$$\mathcal{P}'_P : \begin{cases} \displaystyle\max_{t_0,\, t_1,\, \theta_I;\, t_0 \le t_1} & \min\left\{\tfrac{1}{2}\left(t_0 + t_1\right) - k, t_0\right\} \\ s.t. \ (\mathrm{BR}), \ (\mathrm{IC}), \ (L_1), \ \text{and} \ (U_0). \end{cases}$$

We can now characterize the intermediary's optimal policy by comparing the values associated with the programs $\mathcal{P}_F$ with $\mathcal{P}'_P$.

Proposition 4. (**Optimal disclosure policy with public signal**) There exist thresholds $\overline{\pi}$ and $\pi^{FD}(\overline{\pi} < \pi^{FD})$ such that:

(i) If $\pi \le \overline{\pi}$, the optimal policy is the same as that in the case of a completely uninformative public signal: Partial disclosure is optimal iff $k > k'$ (where k' is as defined in Proposition 2).

(ii) If $\pi \in (\overline{\pi}, \pi^{FD}]$, there exists an interval $(k_1(\pi), k_2(\pi))$, where $k' < k_1(\pi) \le k^* \le k_2(\pi)$, such that the optimal policy calls for no disclosure if $k \in (k_1(\pi), k_2(\pi))$ and full disclosure otherwise. Moreover, the interval shrinks with π.

(iii) If $\pi > \pi^{FD}$, full disclosure is optimal for all k.

Proposition 4 illustrates how the optimal disclosure policy varies with the entry cost and the precision of the public signal. When the public signal is relatively imprecise, the optimal disclosure policy is identical to its counterpart in Section 4—partial disclosure is optimal if the entry cost is sufficiently high and full disclosure is optimal otherwise.

When the public signal is relatively precise, full disclosure is optimal if the entry cost is either too high or too low, and for the moderate values of the entry cost, the optimal disclosure policy calls for full opacity—that is, the intermediary's signal is pure noise. Moreover, this range of entry cost (i.e., where no disclosure is optimal) gets smaller as the public signal becomes more informative; when the public signal is sufficiently precise, full disclosure is optimal regardless of the cost of entry.

The intuition for this result is as follows: Recall that the intermediary's program $\mathcal{P}_P$ (i.e., when public signal is uninformative) is a relaxed version of $\mathcal{P}'_P$. If the solution to $\mathcal{P}_P$ is within the bounds (L_1) and (U_0), it is also a solution to $\mathcal{P}'_P$. This is exactly the case when the public signal is hardly informative (i.e., π is sufficiently small). When π is small, both $\underline{t}_1(\theta_I)$ and $\overline{t}_0(\theta_I)$ are close to the prior expected value of the product, $v(\theta_I)$, and hence, (L_1) and (U_0) are easily satisfied as long as the intermediary's report contains some information on quality (i.e., the buyers' bids t_1 and t_0 upon receiving the intermediary's report would diverge from $\underline{t}_1(\theta_I)$ and $\overline{t}_0(\theta_I)$—their beliefs on quality based on public signal only). Consequently, the optimal disclosure policy would remain qualitatively the same as its counterpart in Section 4.

When the public signal is relatively precise, the bounds (L_1) and (U_0) are tighter and there is less room for the intermediary to garble information. Indeed, when (L_1) and (U_0) bind, the intermediary's optimal partial disclosure policy tantamount to "no disclosure" as its signal no longer contains any information on the product quality. Moreover, when (L_1) and (U_0) are already binding, the intermediary's payoff in $\mathcal{P}'_P$ reduces with the precision of the public signal. As π increases, $\underline{t}_1(\theta_I)$ increases but $\overline{t}_0(\theta_I)$ goes down. Therefore, even though the intermediary sends a completely uninformative signal, it still has to reduce its certification fee to meet the (IR_E) and the (IR_C) constraints—that is, to induce the seller to enter and certify his product.[12] As a result, partial disclosure policy (with full market coverage) becomes less profitable, and the range of entry cost for which no disclosure is optimal shrinks.

Notice that when the public signal is informative, full disclosure can be optimal not only when the entry cost is sufficiently low, but also when it is sufficiently high. As discussed earlier,

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[12] Even if the intermediary does not provide any information whatsoever, it still has a demand for its service (this observation is reminiscent of Lizzeri, 1999). This is due to the fact that a seller who does not use the intermediary is believed to offer a low quality product.

FIGURE 1

OPTIMAL DISCLOSURE POLICY AS A FUNCTION OF k AND π
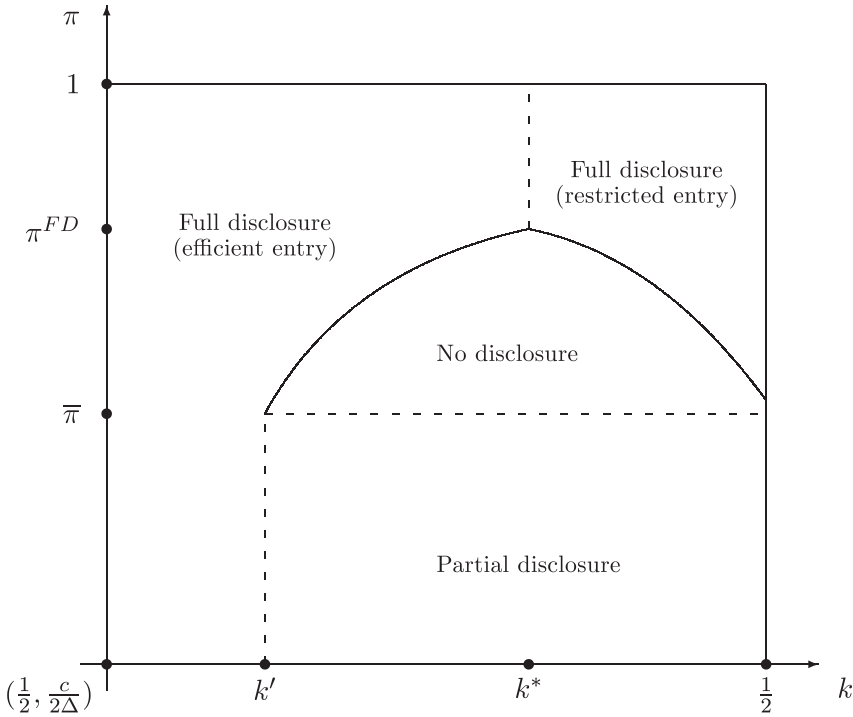
when the entry cost is low, the intermediary can easily accommodate full entry with only a modest reduction in its certification price. However, when the entry cost is sufficiently large, under the optimal partial disclosure policy, accommodating full entry would require a significant price concession (as (IR_E) gets tighter). In contrast, under full disclosure the optimal certification price remains high as entry is restricted (Proposition 1), and the associated payoff of the intermediary is larger than its counterpart under partial disclosure. Figure 1 above depicts a generic example of how the optimal disclosure policy varies with the entry cost and the precision of the public signal.

Proposition 4 has two key implications. First, in the presence of an informative public signal, the optimal certification policy can indeed foreclose the market for some types of the seller; in the absence of the public signal this is never the case. As we will see later, this observation has important implications on social welfare.

Second, in equilibrium, the informativeness of the public signal and that of the intermediary's report exhibit an interesting interplay.

Proposition 5. (**Interplay between public and intermediary's signals**) The optimal certification policy can be implemented by the signal structure given in Proposition 3. Moreover, for any given $k > k'$ (where k' is as defined in Proposition 2) there exists a threshold of π (depending on k), π_k^{FD}, such that:

$$
\rho = \begin{cases} \rho(\pi) & \textit{if } \pi \leq \bar{\pi} \\ 1 & \textit{if } \pi \in (\bar{\pi}, \pi_k^{FD}] \\ 0 & \textit{if } \pi > \pi_k^{FD} \end{cases},
$$

and $\rho(\pi) \in (0, 1]$ is strictly increasing in π.

The above finding indicates that as the public signal becomes more precise, the intermediary's signal becomes less informative, but if the public signal becomes too precise, partial disclosure ceases to be optimal and the intermediary's signal becomes perfectly informative (as the intermediary resorts to full disclosure). In other words, when the public signal is relatively noisy, in the optimal policy, the informativeness of the intermediary's signal behaves as a substitute to that of the public signal. However, if the public signal becomes sufficiently precise, they become complements as the intermediary opts for full disclosure.

To see the reasoning, consider a $k > k'$ so that the optimal policy would call for partial disclosure if the public signal were fully uninformative. Now, when π increases but is still small, the bounds (L_1) and (U_0) remain slack. Hence, in equilibrium, the seller with quality v obtains the same expected bid (t_v) that he would have obtained in the absence of the public signal. In other words, when the public signal gets more precise, the intermediary continues to filter information so that the expected bids received by both the low- and the high-quality sellers remain unaltered. The more precise the public signal, the more the intermediary needs to garble its own report so as to keep the buyers' beliefs unchanged. Thus, ρ increases with π.

In contrast, when the public signal is sufficiently precise (i.e., if $\pi \geq \overline{\pi}$), at the optimal partial disclosure policy, both (L_1) and (U_0) bind. As long as π is moderately large (given the cost of entry), the optimal policy calls for no disclosure, and the intermediary sends the same signal irrespective of the product quality, that is, we have $\rho = 1$, but as π increases further, the optimal policy eventually switches from no disclosure to full disclosure where $\rho = 0$.

6. Welfare

■ How does the presence of the intermediary affect the social welfare in terms of aggregate surplus? Also, how does the aggregate surplus vary as the public signal gets more precise? As all players are risk neutral and there are no inefficiencies at the trading stage of the game (the product is always traded irrespective of the quality), the aggregate surplus in equilibrium only depends on the extent of entry and investment efficiencies (or lack thereof). Thus, the efficiency implications of the intermediary are driven by the entry and investment cutoffs that the optimal policy implements. Also, the value of the intermediary in terms of the social welfare can be ascertained by comparing these cutoffs against their counterpart in the absence of the intermediary.

Let W_I and W_{NI} be the aggregate surplus in equilibrium with and without the intermediary, respectively. The proposition below characterizes the welfare implications of the intermediary's optimal policy.[13]

Proposition 6. (**Welfare implications of the intermediary**) The value of the intermediary, $W_I - W_{NI}$, is (weakly) decreasing in the public signal's precision (π). However, the welfare under intermediary, W_I, is non-monotonic in π. In particular, W_I may decrease in π when π is in a moderate range and the entry cost k is relatively large.

To see the intuition, it is instructive to consider the latter part of this result first. Recall from Proposition 4 that if π is too small or too large, the optimal disclosure policy (in terms of the bids t_0 and t_1 that it induces) is invariant to π. In the former case, partial disclosure is used if and only if the cost of entry is larger than the cutoff k', and in the latter case, the intermediary always uses full disclosure. Consequently, there is no change in welfare.

For moderate values of π, range of entry cost k over which no-disclosure is optimal—that is, (k_1, k_2)—shrinks in π. Again, there is no change in welfare for values of k that are outside this range as there is no change in the intermediary's disclosure policy for such parameters, but for values of k in (k_1, k_2), there could be different welfare consequences based on the value of k.

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[13] For expositional clarity, we only present the salient qualitative nature of the welfare functions. A more formal version of the result is given in the Appendix along with the proof.

If k is relatively small (i.e., closer to k_1), the intermediary's policy switches from no disclosure to full disclosure and does not thwart entry. So welfare increases as entry remains efficient and investment incentives improve. But for k relatively large (i.e., closer to k_2), the policy switches from no disclosure to full disclosure with restricted entry. Though investment incentives improve, the entry inefficiencies outweigh the welfare gains from larger investments, and as a result, the aggregate surplus decreases.

Next, consider the first part of Proposition 6. This statement confirms the common intuition—there is less room for the intermediary to improve investment efficiency when the public signal is already very precise (and entry is always efficient in the absence of the intermediary). However, this argument is incomplete. This argument clearly holds when W_I is non-increasing in π as W_{NI} always increases in π (due to sharper investment incentives), but for values of k where W_I is also increasing in π, it is *a priori* unclear how the difference $W_I - W_{NI}$ would change.

Nevertheless, it can be shown that the gains in W_I is (weakly) less than that in W_{NI}. In particular, for values of k where the intermediary's policy switches from no disclosure to full disclosure (when π increases), the cutoff type for investment remains larger than its counterpart in the absence of intermediary, and for values of k where partial disclosure remains optimal, the investment cutoffs are the same with and without the intermediary.

In this context, two remarks are in order. First, the above argument implies that there is a threshold for the public signal's precision above which the intermediary's presence may be detrimental to social welfare (particularly, when the entry cost is sufficiently large). Second, our analysis sheds light on the welfare implications of mandatory disclosure where the intermediary, if hired, is obligated to offer full disclosure. Though it may appear that urging the intermediary to be more forthcoming with its information would always increase welfare, Proposition 6 suggests that it need not be the case. Trivially, if the public signal is very precise, that is, if $\pi > \pi^{FD}$, such a mandate has no bite as full disclosure is the optimal policy, but if $\pi < \pi^{FD}$, the mandate would change the disclosure policy from no disclosure to full disclosure when $(k_1(\pi), k_2(\pi))$. Such a change would always enhance the seller's investment incentives, but it would also introduce insufficient entry if $k \in (k^*, k_2(\pi))$ and would lead to a decrease in the social welfare.

7. Conclusion

■ We present a model of a certification intermediary where the intermediary's policy influences the seller's investments in product quality as well as his decision on market entry. In our setting, the presence of an intermediary creates a novel trade-off: It improves the seller's incentives to invest in quality upon entry but mutes his entry incentives *ex ante*. The intermediary faces a canonical monopoly problem where a high certification fee facilitates rent extraction from the seller but reduces the demand for certification service at the first place, due to the distortions in both entry and investment incentives. We argue that a partial disclosure policy may be optimal as it can allow the intermediary to charge a relatively high certification fee without causing a large distortion in entry and investment.

Furthermore, we explore how the intermediary's optimal disclosure policy may interact with a public signal on product quality. A key insight that emerges from our model is that the informativeness of the intermediary's signal varies non-monotonically with the public signal's precision. When the public signal is relatively noisy, the intermediary's disclosure policy behaves as a substitute to the public signal—intermediary's report becomes less informative as the public signal becomes more precise—but when the public signal becomes sufficiently precise, the intermediary's report may complement the public signal as the intermediary resorts to full disclosure. Our model also indicates that under high entry cost, the optimal certification policy with full disclosure may call for restricted entry. As a result, an increase in the precision of the public signal may reduce social welfare. Therefore, in the markets with certification intermediaries commonplace interventions such as mandatory disclosure requirements or provision of additional public

information that are geared toward alleviating the information asymmetry may be counterproductive and should be used with caution.

Appendix

This appendix contains the proofs omitted in the text.

Proof of Proposition 1. Clearly, at the optimum, the intermediary would choose a price below $\bar{p} := 1 - (c + k)/\alpha$, as otherwise the demand for the certification service drops to zero; even the most efficient type (i.e., $\theta = 1$) would not enter the market. Now, by plugging in the values of θ_E and θ_I, the program $\mathcal{P}_F$ can be written as:

$$\max_{p \in (0, \bar{p})} \Pi(p) := p \Pr(v = 1 \mid \theta_E, \theta_I)$$

$$= p \left[\frac{1}{2} (\theta_I - \theta_E) + \alpha (1 - \theta_I)] \right]$$

$$= \begin{cases} \Pi_E(p) := p \left[\frac{1}{2} \frac{c}{(1-p)\Delta} + \alpha \left(1 - \frac{c}{(1-p)\Delta} \right) \right] & \text{if } p \leq p_E \\ \Pi_I(p) := p\alpha \left[1 - \frac{c}{\alpha(1-p)-k} \right] & \text{if } p \in [p_E, \bar{p}). \end{cases}$$

The first-order conditions imply:

$$p_E^* = \arg\max_{p \leq p_E} \Pi_E = \begin{cases} 1 - \sqrt{\frac{c}{\alpha}} & \text{if } k < k_E := \sqrt{\frac{c}{4\alpha}} \\ 1 - 2k & \text{otherwise} \end{cases}, \tag{A1}$$

and

$$p_I^* = \arg\max_{p > p_E} \Pi_I = \max \left\{ 1 - \frac{1}{\alpha} \left(k + \sqrt{c(\alpha - k)} \right), 1 - 2k \right\} \tag{A2}$$

$$= \begin{cases} 1 - \frac{1}{\alpha} \left(k + \sqrt{c(\alpha - k)} \right) & \text{if } k > k_I := \frac{1}{8\Delta^2} \left(-c + \sqrt{c^2 + 16c\alpha\Delta^2} \right) \\ 1 - 2k & \text{otherwise} \end{cases}. \tag{A2}$$

Hence, the associated value functions are:

$$\Pi_E^* = \max_{p \leq p_E} \Pi_E(p) = \begin{cases} \overline{\Pi}_E := \alpha + c - 2\sqrt{c\alpha} & \text{if } k < k_E \\ \underline{\Pi}_E := \alpha (1 - 2k) \left(1 - \frac{c}{2k\alpha} \right) & \text{otherwise} \end{cases}, \tag{A3}$$

and

$$\Pi_I^* = \max_{p > p_E} \Pi_I(p) = \begin{cases} \overline{\Pi}_I := \alpha + c - k - 2\sqrt{c(\alpha - k)} & \text{if } k \geq k_I \\ \underline{\Pi}_I := \alpha (1 - 2k) \left(1 - \frac{c}{2k\Delta} \right) & \text{otherwise} \end{cases}. \tag{A4}$$

As $c < \Delta$ (Assumption 1 (i)), we obtain $k_E < k_I$. The proposition follows from the claim that there exists a unique $k^* \in (k_I, 1/2)$ such that $\Pi_I^* \geq \Pi_E^*$ iff $k \geq k^*$. The proof of this claim is given in the following steps:

Step 1: Recall from Assumption 1 (ii) that $k \in (\frac{c}{2\Delta}, \frac{1}{2})$. As $k \to \frac{1}{2}$, $\Pi_I^* = \overline{\Pi}_I > \Pi_E^* = 0$ (as $k \to \frac{1}{2}$, $\overline{\Pi}_I = (\sqrt{\Delta} - \sqrt{c})^2 > 0$ by Assumption 1 (i)) and as $k \to \frac{c}{2\Delta}$, $\Pi_I^* = \underline{\Pi}_I = 0 < \Pi_E^* = \underline{\Pi}_E$. As both Π_I^*s are continuous functions of k, they must intersect at some $k = k^*$, say.

Step 2: *We have $k^* > k_I$; and at k^*, we have $\Pi_E^* = \underline{\Pi}_E = \overline{\Pi}_I = \Pi_I^*$.* The proof is as follows: Note that $\overline{\Pi}_E > \overline{\Pi}_I \ \forall k \in (\frac{c}{2\Delta}, \frac{1}{2})$. To see this, observe that the above inequality can be simplified as:

$$2\sqrt{c\alpha} < k + 2\sqrt{c(\alpha - k)} \ \forall k \in \left(\frac{c}{2\Delta}, \frac{1}{2} \right).$$

Now, $f(k) := k + 2\sqrt{c(\alpha - k)}$ is an increasing function of k ($f' = 1 - \sqrt{c/(\alpha - k)} > 0$ as $\alpha - k > c$) where $f(0) = 2\sqrt{c\alpha}$. Furthermore, we must have $\overline{\Pi}_I \geq \underline{\Pi}_I \ \forall k \in (\frac{c}{2\Delta}, \frac{1}{2})$ as the left-hand side is the value under unconstrained optimum (notice that by definition of k_I, the equality holds only under $k = k_I$). Combining the two, we get $\overline{\Pi}_E > \overline{\Pi}_I \geq \underline{\Pi}_I \ \forall k \in (\frac{c}{2\Delta}, \frac{1}{2})$.

As we know Π_E^* intersects Π_I^* at k^*, it must be the case that at k^*, $\Pi_E^* = \underline{\Pi}_E = \Pi_I^*$. So, $k^* \in (k_E, 1/2)$. Now, recall that $k_E < k_I$, and so, for all $k \in (k_E, k_I]$, $\Pi_E^* = \underline{\Pi}_E > \underline{\Pi}_I = \Pi_I^*$ (as $\alpha > \Delta$). As we know $\underline{\Pi}_E$ intersects Π_I^* at k^*, it must be the case that $k^* > k_I$ and at k^*, we have $\Pi_E^* = \underline{\Pi}_E = \overline{\Pi}_I = \Pi_I^*$.

Step 3: *It must be that k^* is unique.* Recall that at $k = k_I$, $\underline{\Pi}_E > \overline{\Pi}_I = \underline{\Pi}_I$ (as $\Delta < \alpha$) and at $k = 1/2$, $\underline{\Pi}_E < \overline{\Pi}_I$. Moreover $\underline{\Pi}_E$ is concave in k ($\partial^2 \underline{\Pi}_E / \partial k^2 = -c/k^3 < 0$) and $\overline{\Pi}_I$ is convex in k ($\partial^2 \overline{\Pi}_I / \partial k^2 = \sqrt{c}/(2(\alpha - k)^{3/2}) > 0$). Hence k^* must be unique. The proof is by contradiction. Suppose $\underline{\Pi}_E = \overline{\Pi}_I$ at multiple $k \in (k_I, 1/2)$ and let k' and k'' be the smallest and the largest solutions. As $\underline{\Pi}_E(k) > \overline{\Pi}_I(k)$ for $k \in (k_I, k')$, and $\underline{\Pi}_E(k) < \overline{\Pi}_I(k)$ for $k \in (k'', 1/2)$, we have

$$\underline{\Pi}_E'(k') < \overline{\Pi}_I'(k') \text{ and } \underline{\Pi}_E'(k'') > \overline{\Pi}_I'(k'').$$

but as $\underline{\Pi}_E$ is concave in k, we have $\underline{\Pi}'_E(k') > \overline{\Pi}'_E(k'')$. So, we must have

$$\overline{\Pi}'_I(k') > \underline{\Pi}'_E(k') > \overline{\Pi}'_E(k'') > \overline{\Pi}'_I(k''),$$

but this inequality contradicts the fact that $\overline{\Pi}_I$ is convex in k (as we must have $\overline{\Pi}'_I(k') < \overline{\Pi}'_I(k'')$). $\quad\square$

Proof of Lemma 1. It is instructive to argue part (ii) first. This claim follows directly from the fact that both the low and high quality sellers must find it sequentially rational to use the intermediary if pooling between qualities is feasible at the first place.

We prove part (i) by contradiction. We argue that if the optimal partial disclosure policy excludes some types from entry, the resulting payoff is less than that under full disclosure.

Step 1: In any equilibrium where only some types enter, it must be the case that entry is profitable only if the seller invests. Now, we already argued that both types must use the intermediary in equilibrium. So, the intermediary's price cannot exceed t_0. Hence, for any entry cutoff θ_I, the intermediary's payoff is $(1 - \theta_I)t_0$. Moreover, the marginal type must be willing to enter, whereas it must be unprofitable to enter if the firm does not undertake any investment. Hence, the optimal policy must satisfy the following constraints: (i) Bayes Rationality (BR) : $\alpha t_1 + (1 - \alpha)t_0 = \alpha$; (b) Entry for marginal type (E_θ) : $\alpha - k - c/\theta_I \geq t_0$; (c) Essentiality of investment (E_I) : $t_0 \geq \frac{1}{2}(t_0 + t_1) - k$. Hence, the intermediary's problem is:

$$\mathcal{P} : \max_{t_0, t_1, \theta_I} \tilde{\Pi} := (1 - \theta_I)t_0 \ \ s.t. \ (\text{BR}), \ (E_\theta), \ (E_I).$$

Step 2: Consider a relaxed problem $\mathcal{P}' : \max_{t_0, \theta_I} \tilde{\Pi}_R := (1 - \theta_I)t_0$ s.t. (E_θ). The solution to $\mathcal{P}'$:

$$\theta_I = \sqrt{\frac{c}{\alpha - k}} \text{ and } t_0 = \alpha - k - \sqrt{c(\alpha - k)},$$

and the value of the problem is:

$$\tilde{\Pi}^*_R = \alpha + c - k - 2\sqrt{c(\alpha - k)}.$$

Step 3: From (BR), we have $t_1 = 1 - \frac{1-\alpha}{\alpha}t_0$. So, from (E_I), we obtain that the solution to the relaxed problem $\mathcal{P}'$ is feasible in the general problem $\mathcal{P}$ if $k\Delta \geq \frac{1}{2}\sqrt{c(\alpha - k)} \Leftrightarrow k \geq k_I$. Let $\tilde{\Pi}^*$ be the value of problem $\mathcal{P}$. Thus, we have $\tilde{\Pi}^* \leq \tilde{\Pi}^*_R$ for all k, with the equality holding iff $k \geq k_I$.

Step 4: Finally, we argue that for all k, $\tilde{\Pi}^*_R \leq \Pi^*$, the intermediary's payoff under full disclosure, with equality holding iff $k \geq k^*$.

First, notice that from equation (A4) we have $\tilde{\Pi}^*_R = \overline{\Pi}_I$, and from the proof of Proposition 1, we know that $\Pi^* = \overline{\Pi}_I$ for $k \geq k^* > k_I$. Combining these two observations, we obtain that $\tilde{\Pi}^*_R = \Pi^*$ for $k \geq k^*$.

Second, also from the proof of Proposition 1, we know that for $k < k^*$, $\Pi^* = \min\{\overline{\Pi}_E, \underline{\Pi}_E\}$. Recall that $\Pi^* = \overline{\Pi}_E$ for $k < k_E$ and $\overline{\Pi}_E$ is constant in k, and for $k_E \leq k < k^*$, $\Pi^* = \overline{\Pi}_E$, where $\underline{\Pi}_E$ is decreasing in k and intersects $\overline{\Pi}_I$ from above at k^*. Now, it is routine to check that $\overline{\Pi}_I < \overline{\Pi}_E$ for all $k > 0$, and $\overline{\Pi}_I < \underline{\Pi}_E$ at $k = k_E$. As $\overline{\Pi}_E$ is concave and $\overline{\Pi}_I$ is convex in k (step 3 in proof of Proposition 1), it follows that $\overline{\Pi}_I < \Pi^*$ for all $k < k^*$.

Thus, we have $\tilde{\Pi}^* \leq \Pi^*$ for all k, with the equality holding iff $k \geq k^*$. Hence, if partial disclosure is strictly optimal, it must be the case that all types enter. $\quad\square$

Proof of Propositions 2 and 3. These results follow as special cases (where $\pi = \frac{1}{2}$) of Proposition 4 and Proposition 5, respectively. $\quad\square$

The proof of Proposition 4 relies on the following lemma:

Lemma 2. (**Optimal partial disclosure policy under full market coverage**) The solution to $\mathcal{P}'_P$, $(\tilde{t}_0, \tilde{t}_1, \tilde{\theta}_I)$, is characterized as follows:

(i) The optimal cutoff above which all types of the seller invest in quality is given by

$$\tilde{\theta}_I = \min\{\hat{\theta}_I(k), \ \bar{\theta}_I(\pi)\},$$

where $\hat{\theta}_I$ is independent of π but decreasing in k, and $\bar{\theta}_I$ is independent of k but decreasing in π.

(ii) There exist two thresholds, $\underline{\pi}$ and $\overline{\pi}$, where $\frac{1}{2} < \underline{\pi} < \overline{\pi} < 1$, such that for any given k, $\tilde{\theta}_I = \hat{\theta}_I(k)$ if $\pi < \underline{\pi}$, and $\tilde{\theta}_I = \bar{\theta}_I(\pi)$ if $\pi > \overline{\pi}$. Otherwise, that is, if $\pi \in [\underline{\pi}, \overline{\pi}]$, there exists a threshold $k_\pi \in [\sqrt{\frac{c}{4\alpha}}, \sqrt{\frac{c}{4\Delta}}]$ such that $\tilde{\theta}_I = \bar{\theta}_I(\pi)$ if $k < k_\pi$ and $\tilde{\theta}_I = \hat{\theta}_I(\pi)$ otherwise. Moreover, k_π is increasing in π.

(iii) The seller's prices $\tilde{t}_0$ and $\tilde{t}_1$ solve (BR) and (IC) evaluated at $\tilde{\theta}_I$.

Proof. Part (i): We solve $\mathcal{P}'_P$ in the following two steps:

Step 1: Consider the relaxed problem $\mathcal{P}'_{P,R}$ obtained from $\mathcal{P}'_P$ by ignoring (L_1) and (U_0), and replacing (IC) by

$$\theta_I = \frac{c}{(t_1 - t_0)\Delta} \quad (\text{IC}_R).$$

Step 1a: Using (IC_R) and (BR) constraints to solve for t_is, we obtain:

$$t_0(\theta_I) = v(\theta_I) - \frac{c}{\theta_I}\left(\frac{\alpha}{\Delta} - \theta_I\right), \quad t_1(\theta_I) = v(\theta_I) + \frac{c}{\theta_I}\left(\frac{1-\alpha}{\Delta} + \theta_I\right), \tag{A5}$$

where $v(\theta_I) = \alpha(1 - \theta_I) + \frac{1}{2}\theta_I$. So, $\mathcal{P}'_{P,R}$ boils down to:

$$\max_{\theta_I \in [0,1]} p(\theta_I) = \min\left\{\frac{1}{2}(t_0(\theta_I) + t_1(\theta_I)) - k, t_0(\theta_I)\right\}.$$

Step 1b: Notice the following:

(a) Both $t_0(\theta_I)$ and $t_0(\theta_I) + t_1(\theta_I)$ are concave in θ_I; hence, so is $p(\theta_I)$.

(b) $\frac{1}{2}\{t_0(\theta_I) + t_1(\theta_I)\} - k - t_0(\theta_I)$ is decreasing in θ_I, and the equation $\frac{1}{2}\{t_0(\theta_I) + t_1(\theta_I)\} - k - t_0(\theta_I) = 0$ has a unique root $\frac{c}{2k\Delta}$. Hence,

$$p(\theta_I) = \begin{cases} t_0(\theta_I) & \text{if } \theta_I < \frac{c}{2k\Delta} \\ \frac{1}{2}\{t_0(\theta_I) + t_1(\theta_I)\} - k & \text{otherwise} \end{cases}.$$

(c) Finally,

$$\arg\max_{\theta_I} t_0(\theta_I) = \frac{1}{\Delta}\sqrt{c\alpha}, \text{ and } \arg\max_{\theta_I} \frac{1}{2}\{t_0(\theta_I) + t_1(\theta_I)\} - k = \sqrt{\frac{c}{\Delta}}.$$

where $\sqrt{\frac{c}{\Delta}} < \frac{1}{\Delta}\sqrt{c\alpha}$.

Step 1c: The observations (a) to (b) imply the following:

(A) if $\frac{1}{\Delta}\sqrt{c\alpha} < \frac{c}{2k\Delta}$, that is, if $k < \sqrt{\frac{c}{4\alpha}}$,

$$\max p(\theta_I) = \max t_0(\theta_I),$$

(B) if $\frac{c}{2k\Delta} < \sqrt{\frac{c}{\Delta}}$, that is, $k > \sqrt{\frac{c}{4\Delta}}$,

$$\max p(\theta_I) = \max \frac{1}{2}\{t_0(\theta_I) + t_1(\theta_I)\} - k,$$

(C) otherwise,

$$\max p(\theta_I) = t_0\left(\frac{c}{2k\Delta}\right).$$

Hence, the solution to the relaxed problem is as follows:

$$\widehat{\theta}_I = \begin{cases} \frac{1}{\Delta}\sqrt{c\alpha} & \text{if } k < \sqrt{\frac{c}{4\alpha}} \\ \frac{c}{2k\Delta} & \text{if } k \in \left[\sqrt{\frac{c}{4\alpha}}, \sqrt{\frac{c}{4\Delta}}\right] \\ \sqrt{\frac{c}{\Delta}} & \text{if } k > \sqrt{\frac{c}{4\Delta}} \end{cases}. \tag{A6}$$

It is routine to check that $\hat{\theta}_I$ is decreasing in k.

Step 2: Consider the original problem $\mathcal{P}'_P$. Note that the solution to $\mathcal{P}'_{P,R}$ satisfies (IC) as $\frac{1}{\Delta}\sqrt{c\alpha} < 1$ (by Assumption 1 (i)). So, it remains to check when $\hat{\theta}_I$ may violate the constraints (L_1) and (U_0) and what is the solution to $\mathcal{P}'_P$ in such a scenario. Notice that:

$$\underline{t}_1(\theta_I) = \frac{\pi^2 v(\theta_I)}{\pi v(\theta_I) + (1 - \pi)(1 - v(\theta_I))} + \frac{(1 - \pi)^2 v(\theta_I)}{(1 - \pi)v(\theta_I) + \pi(1 - v(\theta_I))}, \tag{A7}$$

and

$$\bar{t}_0(\theta_I) = \frac{\pi(1 - \pi)v(\theta_I)}{(1 - \pi)v(\theta_I) + \pi(1 - v(\theta_I))} + \frac{(1 - \pi)\pi v(\theta_I)}{\pi v(\theta_I) + (1 - \pi)(1 - v(\theta_I))}, \tag{A8}$$

where $v(\theta_I) = \frac{1}{2}\theta_I + \alpha(1 - \theta_I)$, as defined earlier.

Step 2a: For $\theta_I \leq \frac{1}{\Delta}\sqrt{c\alpha}$, it is routine to check that (i) $t_1(\theta_I) - \underline{t}_1(\theta_I)$ is (strictly) decreasing and $t_0(\theta_I) - \bar{t}_0(\theta_I)$ is (strictly) increasing in θ_I. (ii) $\underline{t}_1(\theta_I)$ is increasing in π and for all θ_I, $\underline{t}_1(\theta_I) \to 1$ as $\pi \to 1$. (iii) $\underline{t}_0(\theta_I)$ is decreasing in π and for all θ_I, $\bar{t}_0(\theta_I) \to 0$ as $\pi \to 1$.

Step 2b: Note that $t_0(\theta_I) \to -\infty$ and $t_1(\theta_I) \to \infty$ as $\theta_I \to 0$. Hence, using the observations (i) to (iii) in *Step 2a* above, we can claim the following:

(I) Either $t_1(\theta_I) \geq \underline{t}_1(\theta_I)$ for all $\theta_I \in (0, \frac{1}{\Delta}\sqrt{c\alpha}]$, or there exists a unique $\theta \in (0, \frac{1}{\Delta}\sqrt{c\alpha}]$, θ_I^1 (say), such that $t_1(\theta_I^1) = \underline{t}_1(\theta_I^1)$ and $t_1(\theta_I) \geq \underline{t}_1(\theta_I)$ iff $\theta_I \leq \theta_I^1$. Moreover, θ_I^1 is continuous and decreasing in π as $\underline{t}_1(\theta_I)$ is increasing in π for all θ_I whereas $t_1(\theta_I)$ is independent of π.

(II) Either $t_0(\theta_I) \leq \bar{t}_0(\theta_I)$ for all $\theta_I \in (0, \frac{1}{\Delta}\sqrt{c\alpha}]$, or there exists a unique $\theta \in (0, \frac{1}{\Delta}\sqrt{c\alpha}]$, θ_I^0 (say), such that $t_0(\theta_I^0) = \bar{t}_0(\theta_I^0)$ and $t_0(\theta_I) \leq \bar{t}_0(\theta_I)$ iff $\theta_I \leq \theta_I^0$. Moreover, θ_I^0 is continuous and decreasing in π as $\bar{t}_0(\theta_I)$ is decreasing in π for all θ_I whereas $t_0(\theta_I)$ is independent of π.

(III) If θ_I^0 exists, so does θ_I^1, and vice versa. Moreover, it must be that $\theta_I^1 = \theta_I^0$. To see this consider the term

$$\phi(\theta_I) := v(\theta_I)\big(t_1(\theta_I) - \underline{t}_1(\theta_I)\big) + (1 - v(\theta_I))\big(t_0(\theta_I) - \bar{t}_0(\theta_I)\big).$$

Notice that by (BR),

$$\phi(\theta_I) = \quad v(\theta_I) - \big[v(\theta_I)\underline{t}_1(\theta_I) + (1 - v(\theta_I))\bar{t}_0(\theta_I)\big]$$

$$= \quad v(\theta_I) - \big\{v(\theta_I)\mathbb{E}_{z|v=1}\mathbb{E}_v[v \mid z, \Theta_E = [0, 1], \ \Theta_I = [\theta_I, 1]]$$
$$+ (1 - v(\theta_I))\mathbb{E}_{z|v=0}\mathbb{E}_v[v \mid z, \Theta_E = [0, 1], \ \Theta_I = [\theta_I, 1]]\big\}$$

$$= \quad v(\theta_I) - v(\theta_I) = 0.$$

Hence, if θ_I^0 exists such that $t_0(\theta_I^0) - \bar{t}_0(\theta_I^0) = 0$, it must be that $t_1(\theta_I^0) - \underline{t}_1(\theta_I^0) = 0$; that is, $\theta_I^0 = \theta_I^1$.

Step 2c: Define

$$\bar{\theta}_I = \begin{cases} \theta_I^0 & \text{if exists} \\ \frac{1}{\Delta}\sqrt{c\alpha} & \text{otherwise} \end{cases}. \tag{A9}$$

As $p(\theta_I)$ is concave, if $\hat{\theta}_I$ is not feasible under (L_1) and (U_0), $p(\theta_I)$ is maximized at the largest feasible $\theta_I < \hat{\theta}_I$. That is, the solution to $\mathcal{P}'_P$ is:

$$\tilde{\theta}_I = \min\{\hat{\theta}_I, \bar{\theta}_I\}. \tag{A10}$$

Finally, $\bar{\theta}_I$ is decreasing in π as $\theta_I^0 (= \theta_I^1)$ is.

This observation completes the proof if part (i).

Part (ii): For $\pi = \frac{1}{2}$, $t_0(\theta_I) < \bar{t}_0(\theta_I) = v(\theta_I)$ and $t_1(\theta_I) > \underline{t}_1(\theta_I) = v(\theta_I)$ for all θ_I. Hence, from observations in *step 2a* above, we can claim that there exists a $\underline{\pi}$ such that for all $\pi < \underline{\pi}$, $\theta_I^0 (= \theta_I^1)$ does not exist and therefore, $\bar{\theta}_I = \frac{1}{\Delta}\sqrt{c\alpha}$. So, $\tilde{\theta}_I = \hat{\theta}_I$ as $\frac{1}{\Delta}\sqrt{c\alpha} \geq \hat{\theta}_I$ (by A6). Similarly, as $\theta_I^0 (= \theta_I^1)$ is continuous and decreasing in π, there exists a $\overline{\pi} > \underline{\pi}$ such that for all $\pi > \overline{\pi}$, $\bar{\theta}_I = \theta_I^0 < \sqrt{\frac{c}{\Delta}}$, and hence, $\tilde{\theta}_I = \bar{\theta}_I$. For $\pi \in [\underline{\pi}, \overline{\pi}]$, $\bar{\theta}_I \in [\sqrt{\frac{c}{\Delta}}, \frac{1}{\Delta}\sqrt{c\alpha}]$. As $\hat{\theta}_I$ is decreasing in k, for the expression for $\hat{\theta}_I$ we obtain that there exists a $k_\pi \in [\sqrt{\frac{c}{4\alpha}}, \sqrt{\frac{c}{4\Delta}}]$ such that $\hat{\theta}_I \leq \bar{\theta}_I$ iff $k \geq k_\pi$. Hence, $\tilde{\theta}_I = \min\{\hat{\theta}_I, \bar{\theta}_I\} = \bar{\theta}_I$ iff $k < k_\pi$. Finally, as $\hat{\theta}_I$ is decreasing in k (but independent of π) whereas $\bar{\theta}_I$ is independent of k but decreasing in π, k_π must be increasing in π.

Part (iii): This observation directly follows from the fact at any feasible solution to $\mathcal{P}'_P$, (BR) and (IC) must hold. $\qquad \square$

Proof of Proposition 4. **Step 1**. It follows from Proposition 1 (using equations (A3) and (A4)) that the intermediary's payoff under full disclosure is given as:

$$\Pi^* = \begin{cases} \overline{\Pi}_E = \alpha + c - 2\sqrt{c\alpha} & \text{if } k < \sqrt{\frac{c}{4\alpha}} \\ \underline{\Pi}_E = \alpha(1 - 2k)\left(1 - \frac{c}{2\alpha k}\right) & \text{if } k \in \left[\sqrt{\frac{c}{4\alpha}}, k^*\right] \\ \overline{\Pi}_I = \alpha + c - 2\sqrt{c(\alpha - k)} - k & \text{if } k > k^* \end{cases}. \tag{A11}$$

Step 2. If $\pi < \overline{\pi}$, by Lemma 2 we know that there exists a $k_\pi \in [\sqrt{\frac{c}{4\alpha}}, \sqrt{\frac{c}{4\Delta}}]$ such that the solution to the program $\mathcal{P}'_P$ yields:

$$\tilde{\theta}_I = \begin{cases} \bar{\theta}_I & \text{if } k < k_\pi \\ \frac{c}{2k\Delta} & \text{if } k \in \left[k_\pi, \sqrt{\frac{c}{4\Delta}}\right] \\ \sqrt{\frac{c}{\Delta}} & \text{if } k > \sqrt{\frac{c}{4\Delta}} \end{cases}, \tag{A12}$$

As $\bar{\theta}_I = \hat{\theta}_I(k_\pi) = \frac{c}{2k_\pi \Delta}$, $\tilde{\theta}_I$ is continuous. Let Π_P be the value associated with the program $\mathcal{P}'_P$. So, we have:

$$\Pi_P(k) = \begin{cases} \alpha(1 - 2k_\pi)\left(1 - \frac{c}{2\alpha k_\pi}\right) & \text{if } k < k_\pi \\ \alpha(1 - 2k)\left(1 - \frac{c}{2\alpha k}\right) & \text{if } k \in \left[k_\pi, \sqrt{\frac{c}{4\Delta}}\right] \\ \alpha + c - 2\sqrt{c\Delta} - k & \text{if } k > \sqrt{\frac{c}{4\Delta}} \end{cases}.$$

Now, recall that $\underline{\Pi}_E - \bar{\Pi}_I$ is decreasing in k, strictly positive at $k = \sqrt{\frac{c}{4\Delta}}$, but 0 at k^*. So $k^* > \sqrt{\frac{c}{4\Delta}}$. So, from (A11) and (A12) it readily follows that $\Pi_P(k) > \Pi^*$ iff $k > k' := \sqrt{\frac{c}{4\Delta}}$.

(This case subsumes the case of $\pi = \frac{1}{2}$, and hence, proves Proposition 2.)

Step 3. If $\pi > \bar{\pi}$, by Lemma 2 (part (ii)), we have $\tilde{\theta}_I = \bar{\theta}_I < \sqrt{\frac{c}{\Delta}}$. So,

$$\Pi_P(k) = \begin{cases} \alpha\left(1 - 2k_1\right)\left(1 - \frac{c}{2\alpha k_1}\right) & if\ k \le k_1 \\ \frac{1}{2}\left\{t_0\left(\bar{\theta}_I\right) + t_1\left(\bar{\theta}_I\right)\right\} - k & if\ k > k_1 \end{cases},$$

where

$$k_1 := \frac{c}{2\bar{\theta}_I \Delta}, \quad or,\ equivalently, \ \bar{\theta}_I = \frac{c}{2k_1 \Delta}. \tag{A13}$$

Notice that $\Pi_P(k)$ is continuous and decreasing in k, as, by definition of k_1, we have:

$$\alpha(1 - 2k_1)\left(1 - \frac{c}{2\alpha k_1}\right) = t_0(\bar{\theta}_I) = \frac{1}{2}\left\{t_0(\bar{\theta}_I) + t_1(\bar{\theta}_I)\right\} - k_1.$$

Step 4. Now, if $k_1 < k^*$, we have

$$\alpha(1 - 2k_1)\left(1 - \frac{c}{2\alpha k_1}\right) = \frac{1}{2}\left\{t_0(\bar{\theta}_I) + t_1(\bar{\theta}_I)\right\} - k_1 > \alpha + c - 2\sqrt{c(\alpha - k_1)} - k_1.$$

At $k = \frac{1}{2}$, we must also have

$$\frac{1}{2}\left\{t_0(\bar{\theta}_I) + t_1(\bar{\theta}_I)\right\} - k < \alpha + c - 2\sqrt{c(\alpha - k)} - k.$$

To see this, notice that

$$\frac{1}{2}\left\{t_0(\bar{\theta}_I) + t_1(\bar{\theta}_I)\right\} < \frac{1}{2}\left\{t_0\left(\sqrt{\frac{c}{\Delta}}\right) + t_1\left(\sqrt{\frac{c}{\Delta}}\right)\right\},$$

as $t_0(\theta_I) + t_1(\theta_I)$ is strictly increasing in θ_I for $\theta_I < \sqrt{\frac{c}{\Delta}}$ and $\bar{\theta}_I < \sqrt{\frac{c}{\Delta}}$. Also,

$$\frac{1}{2}\left\{t_0\left(\sqrt{\frac{c}{\Delta}}\right) + t_1\left(\sqrt{\frac{c}{\Delta}}\right)\right\} = \alpha + c - 2\sqrt{c(\alpha - k)}\ when\ k = \frac{1}{2}.$$

So, there exists a cutoff k_2 where $k_2 > k^* > k_1$ and is the unique solution to

$$\frac{1}{2}\left\{t_0(\bar{\theta}_I) + t_1(\bar{\theta}_I)\right\} = \alpha + c - 2\sqrt{c(\alpha - k)}, \tag{A14}$$

such that $\Pi_P(k) > \Pi^*$ iff $k \in (k_1, k_2)$.

Step 5. Note that at $\pi = \frac{1}{2}$, $\bar{\theta}_I = \sqrt{\frac{c}{\Delta}}$. So, $k_1 = \sqrt{\frac{c}{4\Delta}}$ and $k_2 = \frac{1}{2}$. As $\bar{\theta}_I$ is decreasing in π, (A13) and (A14) imply that k_1 is increasing and k_2 is decreasing in π. So (k_1, k_2) shrinks in π.

Furthermore, as $\pi \to 1$, $\Pi_P(k) \to 0$ for any k (as $t_0 \to 0$), and hence, $k_1 \to \frac{1}{2}(> k^*)$. So, there exists a threshold for π, π^{FD} (say) such that for $\pi > \pi^{FD}$, $k_1 > k^*$, and therefore, $\Pi_P(k) < \Pi^*$ for all k. $\qquad\square$

Proof of Proposition 5. **Step 1**: Consider any equilibrium where partial disclosure is optimal and the solution to the intermediary's problem is $(\tilde{t}_0, \tilde{t}_1, \tilde{\theta}_I)$. Notice that:

$$\begin{aligned} t_1(\theta_I) = \ & \mathbb{E}_{(x,z)|v=i}\mathbb{E}_v\big[v \mid x,\ z,\ \Theta_E = [0,1],\ \Theta_I = [\tilde{\theta}_I, 1]\big] \\[4pt] = \ & \sum_{x,z} \Pr[v = 1 \mid x,\ z,\ \tilde{\theta}_I] \times \Pr[x,\ z \mid v = 1] \\[4pt] = \ & \Pr[v = 1 \mid x_1,\ z = 1,\ \tilde{\theta}_I] \times \Pr[x_1,\ z = 1 \mid v = 1] \\ & + \Pr[v = 1 \mid x_1,\ z = 0,\ \tilde{\theta}_I] \times \Pr[x_1,\ z = 0 \mid v = 1] \\[4pt] = \ & \Pr[v = 1 \mid x_1,\ z = 1,\ \tilde{\theta}_I]\pi + \Pr[v = 1 \mid x_1,\ z = 0,\ \tilde{\theta}_I](1 - \pi). \end{aligned}$$

Step 2: Now, consider a disclosure policy where the set of signals is $X = \{x_1, x_0\}$ and $\Pr(x_1 \mid v = 1) = 1$ and $\Pr(x_1 \mid v = 0) = \rho$. Under this policy, for a given ρ, we have:

$$\begin{aligned} \Pr\big[v = 1 \mid x_1,\ z = 1,\ \tilde{\theta}_I\big] &= \frac{\Pr\big[x_1,\ z = 1 \mid v = 1,\ \tilde{\theta}_I\big]\Pr\big[v = 1 \mid \tilde{\theta}_I\big]}{\sum_v \Pr\big[x_1,\ z = 1 \mid v,\ \tilde{\theta}_I\big]\Pr\big[v \mid \tilde{\theta}_I\big]} \\[6pt] &= \frac{\pi v(\tilde{\theta}_I)}{\pi v(\tilde{\theta}_I) + \rho(1 - \pi)\big(1 - v(\tilde{\theta}_I)\big)}, \end{aligned}$$

and, similarly,

$$\Pr\big[v = 1 \mid x_1,\ z = 0,\ \tilde{\theta}_I\big] = \frac{(1 - \pi)v(\tilde{\theta}_I)}{(1 - \pi)v(\tilde{\theta}_I) + \rho\pi\big(1 - v(\tilde{\theta}_I)\big)}.$$

Thus, we have

$$t_1(\theta_I) = \frac{\pi^2 v(\tilde{\theta}_I)}{\pi v(\tilde{\theta}_I) + \rho(1-\pi)(1-v(\tilde{\theta}_I))} + \frac{(1-\pi)^2 v(\tilde{\theta}_I)}{(1-\pi)v(\tilde{\theta}_I) + \rho\pi(1-v(\tilde{\theta}_I))},$$

and ρ must solve:

$$\tilde{t}_1 = \frac{\pi^2 v(\tilde{\theta}_I)}{\pi v(\tilde{\theta}_I) + \rho(1-\pi)(1-v(\tilde{\theta}_I))} + \frac{(1-\pi)^2 v(\tilde{\theta}_I)}{(1-\pi)v(\tilde{\theta}_I) + \rho\pi(1-v(\tilde{\theta}_I))}. \tag{A15}$$

Step 3: As ρ takes all values from 0 to 1, and hence the right-hand side takes all values from $\underline{t}_1(\tilde{\theta}_I)$ to 1, where

$$\underline{t}_1(\tilde{\theta}_I) = \mathbb{E}_{z|v=1}\mathbb{E}_v\big[v \mid z, \Theta_E = [0,1], \ \Theta_I = [\tilde{\theta}_I, 1]\big]$$

$$= \frac{\pi^2 v(\tilde{\theta}_I)}{\pi v(\tilde{\theta}_I) + (1-\pi)(1-v(\tilde{\theta}_I))} + \frac{(1-\pi)^2 v(\tilde{\theta}_I)}{(1-\pi)v(\tilde{\theta}_I) + \pi(1-v(\tilde{\theta}_I))}.$$

Hence, for any $\tilde{t}_1 \in [\underline{t}_1(\tilde{\theta}_I), 1]$, there exists a value of ρ that satisfies (A15). The corresponding value of $\tilde{t}_0$ is uniquely pins down by (BR) as

$$\tilde{t}_0 = \frac{v(\tilde{\theta}_I)}{1-v(\tilde{\theta}_I)}(1-\tilde{t}_1).$$

Hence, the optimal disclosure policy can be implemented by a signal structure given in Proposition 3.

Step 4: Fix any $k > k' := \sqrt{\frac{c}{4\Delta}}$. From Proposition 4, we know that for $\pi \le \overline{\pi}$, the optimal disclosure policy calls for partial disclosure. Moreover, part (ii) of Lemma 2 implies that at the optimum, $\tilde{t}_1 = t_1(\sqrt{\frac{c}{\Delta}})$, and hence, a constant with respect to π (for $k > \sqrt{\frac{c}{4\Delta}}$, $\tilde{\theta}_I = \hat{\theta}_I$ for all π and $\hat{\theta}_I = \sqrt{\frac{c}{\Delta}}$ by equation (A6)). Now the right-hand side of (A15) is increasing in π and decreasing in ρ. Hence, as π increases (but $\pi < \overline{\pi}$), ρ must increase to keep $\tilde{t}_1$ constant.

From part (ii) of Proposition 4 we also know that for $\pi \in (\overline{\pi}, \pi^{FD}]$, the optimal disclosure policy calls for no disclosure as long as $k \in (k_1(\pi), k_2(\pi))$, $k_1(\pi) > k'$. Moreover, $(k_1(\overline{\pi}), k_2(\overline{\pi})) = (k', \frac{1}{2})$, both k_1 and k_2 are continuous, and k_1 is strictly increasing in π whereas k_2 is strictly decreasing in π. Hence, for any $k > k'$, there exists a value of $\pi \in (\overline{\pi}, \pi^{FD}]$, π_k^{FD} (say), such that for $\pi \in (\overline{\pi}, \pi_k^{FD}]$, $k \in (k_1(\pi), k_2(\pi))$, and hence no disclosure is optimal. Thus, we must have $t_1(\theta_I) = \bar{t}_0(\theta_I)$ (as given by equation (A8), which is attained by setting $\rho = 1$. Similarly, for $\pi > \pi_k^{FD}$, $k \notin (k_1(\pi), k_2(\pi))$ and full disclosure is optimal. Hence, we must have $\rho = 0$ for $\pi > \pi_k^{FD}$. $\qquad\square$

In order to prove Proposition 6, it is useful to state the proposition more formally:

Proposition 6′: (i) $\partial W_I / \partial \pi \ne 0$ only if $\pi \in [\overline{\pi}, \pi^{FD}]$.

(ii) Suppose that π increases from π' to π'' where $\pi' < \pi^{FD}$ and $\pi'' > \overline{\pi}$. (a) If $\pi'' < \pi^{FD}$, that is, $(k_1(\pi''), k_2(\pi'')) \ne \emptyset$, W_I (strictly) increases if $k \in [k_1(\pi'), k_2(\pi'')]$, decreases if $k \in [k_2(\pi''), k_2(\pi')]$, and remains constant if $k < k_1(\pi')$ or $k > k_2(\pi')$. (b) If $\pi'' > \pi^{FD}$, that is, full disclosure is optimal for all k at π'' as $(k_1(\pi''), k_2(\pi'')) = \emptyset$, W_I (strictly) increases if $k \in [k_1(\pi'), k^*]$, but decreases otherwise.

(iii) $W_I - W_{NI}$ is (weakly) decreasing in π.

Below, we first present the proofs of parts (i) and (ii) of Proposition 6′.

Proof of Proposition 6′. We being by stating a few preliminary observations.

Notice that for any given type cutoff for entry (θ_E) and investment (θ_I), the aggregate surplus is given as:

$$W(\theta_E, \theta_I) = (\theta_I - \theta_E)\left(\frac{1}{2} - k\right) + \int_{\theta_I}^{1}\left(\alpha - \frac{c}{\theta} - k\right)d\theta.$$

Using Propositions 1 and 4, we obtain:

$$\text{if } \pi < \pi^{FD}, W_I = \begin{cases} W(0, \theta_I^*) & \text{if } k \le k_1(\pi) \\ W(\theta_I^*, \theta_I^*) & \text{if } k \ge k_2(\pi) \\ W(0, \tilde{\theta}_I) & \text{otherwise} \end{cases},$$

$$\text{otherwise}, W_I = \begin{cases} W(0, \theta_I^*) & \text{if } k \le k^* \\ W(\theta_I^*, \theta_I^*) & \text{if } k \ge k^* \end{cases},$$

where θ_I^* and $\tilde{\theta}_I$ are the investment type cutoff in the optimal full and partial disclosure policies, respectively (i.e., the cutoffs obtained in the solutions to $\mathcal{P}_F$ with $\mathcal{P}_P'$).

Thus, by equations (A1) and (A2) we have:

$$\theta_I^* = \begin{cases} \theta_I^f := c/\left(1 - p_E^*\right)\Delta = \min\left\{\frac{\sqrt{c\alpha}}{\Delta}, \frac{c}{2k\Delta}\right\} & \text{if } k < k^* \\ \theta_I^i := c/\left(\alpha\left(1 - p_I^*\right) - k\right) = \sqrt{\frac{c}{\alpha - k}} & \text{otherwise} \end{cases}.$$

Also, recall that

$$\tilde{\theta}_I = \min\{\hat{\theta}_I, \bar{\theta}_I\},$$

where $\hat{\theta}_I$ and $\bar{\theta}_I$ are given by equations (A6) and (A9), respectively. We are now ready present the proofs of parts (i) and (ii).

Part (i): Step 1. From the proof of Proposition 4 (step 2), we know that if $\pi \leq \overline{\pi}$, $(k_1(\pi), k_2(\pi)) = (\sqrt{\frac{c}{4\Delta}}, \frac{1}{2})$, and the following holds: (a) for all $k < \sqrt{\frac{c}{4\Delta}}$, full disclosure is optimal and the associated $\theta_E = 0$ and $\theta_I = \theta_I^f$, (b) for all $k \geq \sqrt{\frac{c}{4\Delta}}$, partial disclosure is optimal and the associated $\theta_E = 0$ and $\theta_I = \tilde{\theta}_I$. Moreover, from the proof of Lemma 2 (part (ii)), we know that $\tilde{\theta}_I = \hat{\theta}_I$ for all $k > k_\pi \geq \sqrt{\frac{c}{4\Delta}}$. Since both θ_I^f and $\hat{\theta}_I$ are independent of π, W_I is invariant to π.

Step 2. If $\pi \geq \pi^{FD}$, W_I depends on θ_I^f, θ_I^i, and k^*, all of which are invariant to π; and hence, so is W_I.

Part (ii): Step 1. First consider the case where $\pi'' < \pi^{FD}$, that is, $(k_1(\pi''), k_2(\pi'')) \neq \emptyset$. The proof is obtained by characterizing W_I in each of the following mutually exclusive and exhaustive parameter ranges for k.

Step 1a: For $k < k_1(\pi')$, full disclosure remains as the optimal policy with $\theta_E = 0$ and $\theta_I = \theta_I^f$. Hence, there is no change in W_I.

Step 1b: For $k \in [k_1(\pi'), k_1(\pi''))$, optimal policy switches from partial disclosure where $(\theta_E, \theta_I) = (0, \tilde{\theta}_I)$ to full disclosure where $(\theta_E, \theta_I) = (0, \theta_I^f)$. We claim that for $k \in [k_1(\pi'), k_1(\pi''))$ and $\pi = \pi'$, $\tilde{\theta}_I \geq \theta_I^f$. The argument is as follows.

As $\pi'' < \pi^{FD}$, $k_1(\pi'') < k^*$. So, for any $k \in [k_1(\pi'), k_1(\pi'')) \subseteq [\sqrt{\frac{c}{4\Delta}}, k^*)$, $\theta_I^f = \frac{c}{2k\Delta}$, and hence, strictly decreasing in k, but $\tilde{\theta}_I$ remains constant. If $\pi' \leq \overline{\pi}$, from the proof of Proposition 4, we know that $k_1(\pi') = \sqrt{\frac{c}{4\Delta}}$. Hence, by Lemma 2 part (ii), $\tilde{\theta}_I = \hat{\theta}_I = \frac{c}{2k_1(\pi')\Delta} = \sqrt{\frac{c}{\Delta}}$. Also, if $\pi' > \overline{\pi}$, from Lemma 2 (part (ii)), we have $\tilde{\theta}_I = \bar{\theta}_I = \frac{c}{2k_1(\pi')\Delta}$ (the last equality follows from equation (A13) in the proof of Proposition 4). Hence, for all $k \in [k_1(\pi'), k_1(\pi''))$, $\tilde{\theta}_I = \frac{c}{2k_1(\pi')\Delta} \geq \frac{c}{2k\Delta} = \theta_I^f$.

Step 1c: For $k \in [k_1(\pi''), k_2(\pi'')]$ partial disclosure remains optimum. So, we have $(\theta_E, \theta_I) = (0, \tilde{\theta}_I)$ and $\tilde{\theta}_I = \min\{\sqrt{\frac{c}{\Delta}}, \bar{\theta}_I\}$. Hence, W_I is increasing in π as θ_I is decreasing in π.

Step 1d: Finally, for $k > k_2(\pi'')$, two cases can arise: (i) for $k > k_2(\pi') > k_2(\pi'')$ full disclosure remains as the optimal policy where $(\theta_E, \theta_I) = (\theta_I^i, \theta_I^i)$, and hence there is no change in W_I for a given k. (ii) For $k \in (k_2(\pi''), k_2(\pi')]$ the optimal policy switches from partial disclosure to full disclosure and the associated type cutoffs change from $(\theta_E, \theta_I) = (0, \tilde{\theta}_I)$ to (θ_I^i, θ_I^i), but notice that $W_I(0, \tilde{\theta}_I) \geq W_I(0, \sqrt{\frac{c}{\Delta}}) > W_I(\theta_I^i, \theta_I^i)$. The first inequality follows from the fact that for such k, $\tilde{\theta}_I = \min\{\sqrt{\frac{c}{\Delta}}, \bar{\theta}_I\}$ and W_I is decreasing in θ_I. The second inequality follows as $W_I(0, \sqrt{\frac{c}{\Delta}}) = W_I(\theta_I^i, \theta_I^i)$ at $k = \frac{1}{2}$ (recall that $\theta_I^i = \sqrt{\frac{c}{\alpha-k}}$), and for $k < \frac{1}{2}$, $\frac{\partial}{\partial k} W_I(0, \sqrt{\frac{c}{\Delta}}) = -1$, whereas $\frac{\partial}{\partial k} W_I(\theta_I^i, \theta_I^i) = -1 + \frac{1}{2}(\frac{c}{\alpha-k} + \sqrt{\frac{c}{\alpha-k}}) \in (-1, 0)$.

Step 2. The case where $\pi'' \geq \pi^{FD}$ can be treated exactly as above by setting $k_1(\pi'') = k_2(\pi'') = k^*$.

The proof of part (iii) of Proposition 6′ relies on a set of lemmas given below.

Lemma 3. If there is no intermediary in the market, the equilibrium has the following features:

(i) Entry is always efficient—all types of the seller enter (i.e., $\theta_E = 0$).

(ii) Investment may be inefficient. There exists a threshold π^{NI} (depending on c) such that if $\pi < \pi^{NI}$ no type invests. Otherwise, there exists a unique cutoff θ_I^{NI} such that all types $\theta \geq \theta_I^{NI}$ invest. Moreover, θ_I^{NI} is decreasing in π.

Proof. **Step 1**. Denote

$$v_i(\Theta_E, \Theta_I) = \sum_{z \in \{0,1\}} \mathbb{E}(v \mid z; \Theta_E, \Theta_I) \Pr(z \mid I = i),$$

which is the expected price the seller receives when his investment decision is I and the buyers believe that the set of types that enter is Θ_E and the set of types that invest is Θ_I. Now, for type θ, the expected payoff from entry and investment decision I is:

$$V(I; \theta, \Theta_E, \Theta_I) = \begin{cases} v_1(\Theta_E, \Theta_I) - k - \frac{c}{\theta} & \text{if } I = 1 \\ v_0(\Theta_E, \Theta_I) - k & \text{if } I = 0 \end{cases}.$$

As for any Θ_E and Θ_I, $V(1; \theta, \Theta_E, \Theta_I)$ is increasing in θ whereas $V(0; \theta, \Theta_E, \Theta_I)$ is constant, in any equilibrium, the seller's investment decision must follow a cutoff strategy as the seller invests if and only if $V(1; \theta, \Theta_E, \Theta_I) \geq V(0; \theta, \Theta_E, \Theta_I)$.

Step 2. Let θ' be the investment cutoff type, that is, $\Theta_I = [\theta', 1]$. For brevity of notation, denote

$$v_i(\theta') = v_i(\Theta_E = [0, 1], \Theta_I = [\theta', 1]),$$

and

$$\mathbb{E}(v \mid z; \ \theta') = \mathbb{E}\big(v \mid z; \ \Theta_E = [0, 1], \ \Theta_I = [\theta', 1]\big).$$

We need to show that there exists a unique θ_I^{NI} such that (i) $v_1(\theta_I^{NI}) - v_0(\theta_I^{NI}) = \frac{c}{\theta_I^{NI}}$ and (ii) $V(0; \theta, \Theta_E, \ \Theta_I) = v_0(\theta_I^{NI}) - k > 0$.

Step 3. It is useful to note that $\Pr(z = 1 \mid I) = \sum_{k \in \{0,1\}} \Pr(z = 1 \mid v = k) P(v = k \mid I)$; so,

$$\Pr(z = 1 \mid I = 1) = \pi\alpha + (1 - \pi)(1 - \alpha), \ \text{and} \ \Pr(z = 1 \mid I = 0) = 1/2.$$

Also,

$$\mathbb{E}(v \mid z; \ \theta') = \Pr(v = 1 \mid z; \ \theta') = \frac{\Pr(z \mid v = 1)\Pr(v = 1 \mid \theta')}{\Pr(z \mid \theta')}.$$

We thus have

$$\mathbb{E}(v \mid z = 1; \ \theta') = \frac{\pi(\alpha - \theta'\Delta)}{\pi(\alpha - \theta'\Delta) + (1 - \pi)(1 - \alpha + \theta'\Delta)},$$

and

$$\mathbb{E}(v \mid z = 0; \ \theta') = \frac{(1 - \pi)(\alpha - \theta'\Delta)}{\pi(1 - \alpha + \theta'\Delta) + (1 - \pi)(\alpha - \theta'\Delta)}.$$

Step 4. Note that

$$\frac{\partial}{\partial\theta'} \mathbb{E}(v \mid z = 1; \ \theta') = -\frac{(1 - \pi)\pi\Delta}{(1 - (\pi - \alpha(2\pi - 1) + (2\pi - 1)\theta'\Delta))^2} < 0,$$

and

$$\frac{\partial}{\partial\theta'} \mathbb{E}(v \mid z = 0; \ \theta') = -\frac{\alpha(1 - \pi)\pi\Delta}{(\pi - \alpha(2\pi - 1) + (2\pi - 1)\theta'\Delta)^2} < 0.$$

This implies that both v_1 and v_0 are decreasing functions of θ'. Note that $v_0(1) = \frac{1}{2}$. So, $v_0(\theta') \geq \frac{1}{2} > k$ for any investment cutoff type $\theta' \in [0, 1]$. Hence, all types of the seller enter for any investment cutoff θ'. This observation completes the proof of part (i) of the proposition. To prove part (ii), we proceed as follows.

Step 5. Now we show that there exists a unique type cutoff for investment. Let $\psi(\theta') = v_1(\theta') - v_0(\theta')$ denote the private value of investment in the presence of adverse selection due to asymmetric information about the product's quality. This value depends on the investment cutoff θ'. Using the above expressions, we obtain

$$\frac{\partial}{\partial\theta'}\psi(\theta') = \frac{1}{(D(1 - D))^2}\big[\pi(1 - \pi)(\Delta(2\pi - 1))^2(2\alpha - 1 - 2\theta'\Delta)\big].$$

where D is a linear function of θ'. Note that $\alpha \geq \alpha - \theta'\Delta \geq 1/2$ as $2\alpha - 1 - 2\theta'\Delta \geq 0$. As a result, we have $\frac{\partial}{\partial\theta'}\psi(\theta') \geq 0$.

Step 6. Note that $\psi(\theta') < c/\theta'$ as $\theta' \to 0$. So, if $\psi(1) = (2\pi - 1)^2\Delta < c$, or, equivalently,

$$\pi < \frac{1}{2} + \sqrt{\frac{c}{4\Delta}} =: \pi^{NI},$$

$\psi(\theta') < c/\theta'$ for all θ' (as $\psi' \geq 0$) and none of the types invests. Otherwise, there exists a cutoff type, θ_I^{NI} (say), such that $\psi(\theta_I^{NI}) = c/\theta_I^{NI}$.

Step 7. Finally, note that

$$\frac{\partial}{\partial\pi}\psi(\theta') = \frac{1}{(D(1 - D))^2}A(1 - A)(2\pi - 1)\Delta > 0,$$

where $A = \alpha(1 - \theta') + \frac{1}{2}\theta' \in \left(\frac{1}{2}, 1\right)$. Hence, θ_I^{NI} is decreasing in π. $\qquad\square$

Lemma 4. Suppose that all types enter but only type $\theta > \theta^* \in [0, 1]$ invests. Then,

$$v_1(\theta^*) - v_0(\theta^*) = \Delta\big(\underline{t}_1(\theta^*) - \bar{t}_0(\theta^*)\big).$$

Proof. The proof follows from direct computation of the terms given in the equation above. Denote $\mathbb{E}(v \mid z; \ \theta^*) := \mathbb{E}(v \mid z; \ \Theta_E = [0, 1], \ \Theta_I = [\theta^*, 1])$ and recall that

$$v_i(\theta^*) := \sum_{z \in \{0,1\}} \mathbb{E}(v \mid z; \ \theta^*)\Pr(z \mid I = i), \ i = 0, 1,$$

that is, the expected value of the product after z is realized. Also note that,

$$
\begin{aligned}
\underline{t}_1(\theta^*) - \bar{t}_0(\theta^*) = & \; \mathbb{E}_{z|v=1}\mathbb{E}_v(v \mid z; \; \theta^*) - \mathbb{E}_{z|v=0}\mathbb{E}_v(v \mid z; \; \theta^*) \\
= & \; \big[\mathbb{E}(v \mid z=1; \; \theta^*)\pi + \mathbb{E}(v \mid z=0; \; \theta^*)(1-\pi)\big] \\
& -\big[\mathbb{E}(v \mid z=1; \; \theta^*)(1-\pi) + \mathbb{E}(v \mid z=0; \; \theta^*)\pi\big].
\end{aligned}
$$

Now,

$$
v_1(\theta^*) - v_0(\theta^*) = \sum_{i \in \{0,1\}}\left[\sum_{z \in \{0,1\}}\mathbb{E}(v \mid z; \; \theta^*)\Pr(z \mid I = i)\right],
$$

and

$$
\begin{aligned}
\Pr(z = 1 \mid I = 1) \; = & \sum_{v \in \{0,1\}}\Pr(z = 1 \mid v, I = 1)\Pr(v \mid I = 1) \\
= & \; \pi\alpha + (1-\pi)(1-\alpha),
\end{aligned}
$$

and

$$
\Pr(z = 1 \mid I = 0) = \frac{1}{2}\pi + \frac{1}{2}(1-\pi) = \frac{1}{2}.
$$

So,

$$
\begin{aligned}
v_1(\theta^*) - v_0(\theta^*) = & \sum_{i \in \{0,1\}}\left[\sum_{z \in \{0,1\}}\mathbb{E}(v \mid z; \; \theta^*)\Pr(z \mid I = i)\right] \\
= & \; \mathbb{E}(v \mid z=1; \; \theta^*)\pi\Delta + \mathbb{E}(v \mid z=1; \; \theta^*)\big((1-\pi)(1-\alpha) - \tfrac{1}{2}(1-\pi)\big) \\
& -\mathbb{E}(v \mid z=0; \; \theta^*)(\pi\alpha + (1-\pi)(1-\alpha)) + \mathbb{E}(v \mid z=0; \; \theta^*)\big(\tfrac{1}{2}\pi + \tfrac{1}{2}(1-\pi)\big) \\
= & \; \mathbb{E}(v \mid z=1; \; \theta^*)\pi\Delta - \mathbb{E}(v \mid z=1; \; \theta^*)(1-\pi)\Delta \\
& -\mathbb{E}(v \mid z=0; \; \theta^*)\pi\Delta + \mathbb{E}(v \mid z=0; \; \theta^*)(1-\pi)\Delta \\
= & \; \Delta\big(\underline{t}_1(\theta^*) - \bar{t}_0(\theta^*)\big).
\end{aligned}
$$

Hence the proof.   ☐

Lemma 5. For $\pi \geq \bar{\pi}$ and $k \in [k_1(\pi), k_2(\pi)]$, $\tilde{\theta}_I = \theta_I^{NI}$. Also, for $\pi < \bar{\pi}$ and $k > \sqrt{\frac{c}{4\Delta}}$, $\tilde{\theta}_I = \sqrt{\frac{c}{\Delta}}$.

Proof. For $\pi \geq \bar{\pi}$ and $k \in [k_1(\pi), k_2(\pi)]$, $\tilde{\theta}_I = \bar{\theta}_I$ where for (IC) we have

$$
\Delta\big(\underline{t}_1(\bar{\theta}_I) - \bar{t}_0(\bar{\theta}_I)\big) = \frac{c}{\bar{\theta}_I}.
$$

So, using Lemma 4, we have

$$
v_1(\bar{\theta}_I) - v_0(\bar{\theta}_I) = \Delta\big(\underline{t}_1(\bar{\theta}_I) - \bar{t}_0(\bar{\theta}_I)\big) = \frac{c}{\bar{\theta}_I}. \tag{A16}
$$

Now, θ_I^{NI} is the unique solution to

$$
v_1(\theta) - v_0(\theta) = \frac{c}{\theta}. \tag{A17}
$$

Hence, we must have

$$
\theta_I^{NI} = \bar{\theta}_I = \tilde{\theta}_I.
$$

For $\pi < \bar{\pi}$, $\tilde{\theta}_I = \hat{\theta}_I$, and hence, from equation (A6) we have $\tilde{\theta}_I = \sqrt{\frac{c}{\Delta}}$ for $k > \sqrt{\frac{c}{4\Delta}}$.   ☐

Lemma 6. $\pi^{NI} < \bar{\pi}$.

Proof. By definition, equation (A17), that is,

$$
v_1(\theta) - v_0(\theta) = \frac{c}{\theta},
$$

has a unique solution in $[0,1)$ if and only if $\pi > \pi^{NI}$ and no solution otherwise. Also, by definition, at $\pi = \bar{\pi}$, $\bar{\theta}_I = \sqrt{\frac{c}{\Delta}} < 1$ and $\Delta\big(\underline{t}_1(\bar{\theta}_I) - \bar{t}_0(\bar{\theta}_I)\big) = \frac{c}{\bar{\theta}_I}$. But by equation (A16) as given in the proof of Lemma 5, $\bar{\theta}_I = \sqrt{\frac{c}{\Delta}}$ is also a solution to equation (A17). Since equation (A17) does not have any solution for $\pi \leq \pi^{NI}$, we must have $\bar{\pi} > \pi^{NI}$.   ☐

We are now ready to present the proof of part (iii) of Proposition 6′.

Proof of Proposition 6′. Part (iii). **Step 1**. By Lemma 3, we can write the aggregate surplus in the absence of the intermediary, W_{NI}, as:

$$W_{NI} = \begin{cases} W(0,1) & \text{if } \pi < \pi^{NI} \\ W(0, \theta_I^{NI}) & \text{otherwise} \end{cases}. \tag{A18}$$

Note that W_{NI} is invariant to π if $\pi < \pi^{NI}$ but increases otherwise (as θ_I^{NI} is decreasing in π).

Step 2. For $\pi < \overline{\pi}$, W_I is invariant to π as it depends on either θ_I^f or θ_i^i, and both are independent of π. Hence, for $\pi < \pi^{NI}$, $W_I - W_{NI}$ is constant (note that by Lemma 6, $\pi^{NI} < \overline{\pi}$) but for $\pi \in [\pi^{NI}, \overline{\pi})$, $W_I - W_{NI}$ is strictly decreasing in π.

Next, consider $\pi \in [\overline{\pi}, \pi^{FD})$. If $k \notin (k_1(\pi), k_2(\pi))$, W_I is invariant to π as it only depends on either θ_I^f or θ_I^i. Otherwise, $W_I = W(0, \tilde{\theta}_I) = W(0, \theta_I^{NI}) = W_{NI}$ (by Lemma 5). Hence, in the former case, $W_I - W_{NI}$ is strictly decreasing in π and in the later case $W_I - W_{NI} = 0$ for all such π.

Finally, for $\pi \geq \pi^{FD}$, W_I is still invariant to π as k^* is also independent of π. So, $W_I - W_{NI}$ is strictly decreasing in π. Hence, the proof. $\square$

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