Managing Loyalty in Hierarchical Organizations*

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Abstract

In many hierarchical organizations, middle managers serve an important dual role: relaying decision-relevant information to the headquarters and fostering initiatives among their subordinates. Such a dual role calls for dual loyalty. The manager remains effective only if his loyalty to the top (relative to the bottom) lies in a moderate range, and this range expands with the manager's skill to evaluate his subordinates' initiatives. When the manager's loyalty to the top undermines incentives at the bottom, relational incentive contracts may sustain nuanced communication and decision-making protocols that restore his effectiveness. Our findings highlight how the middle manager's loyalty to the organization interacts with his loyalty to the top management, and how his personal characteristics, i.e., loyalty and skill, affect the management style within the organization.

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1 Introduction

Middle managers are integral to hierarchical organizations. Positioned "between [an organization's] operating core and the apex" (Mintzberg, 1989; p. 98), they channel information flow within the organization and motivate their subordinates to take actions that further the organization's goals. Such a "dual role" is particularly critical in innovative organizations. Innovations often come from frontline employees with deep knowledge of the organization's day-to-day operations, and the headquarters rely on mid-level managers to winnow and elevate promising ideas. But, as innovative activities are costly and inherently risky, the employees may lack initiative for innovation unless they have a middle manager "on their side" who could persuade the top management to support their work.

For example, the success of IBM in maintaining its innovative edge and adapting to changing technological landscapes is largely attributed to the role its middle managers have played over the years as "powerful innovation cheerleaders and coaches" for its frontline workers (Bensaou, 2021). Middle managers were also instrumental in Intel's switch in the 1980s from the memory chip to the microprocessors market. They identified how the company's existing capabilities may be leveraged to foray into the microprocessor market, championed innovation in microprocessor technology, and subsequently persuaded the top management to exit the memory business and focus on microprocessors (Burgelman, 1994, Foss and Klein, 2022). On the same note, the decline in product innovation among top U.S. tech firms like Microsoft, Apple, and Xerox in the 1990s and 2000s may be traced to middle managers losing their ability to encourage innovation, as companies began valuing salespeople over product engineers, reducing the engineers' motivation to innovate. As Steve Jobs, the co-founder of Apple Inc., notes, "[w]hen the sales guys run the company, the product guys don't matter so much, and a lot of them just turn off." (Isaacson, 2011; p. 569.)

However, to successfully carry out the dual role of fostering innovation among subordinates and conveying crucial information to senior leadership, a middle manager must maintain credibility across all levels of the organization. And, his effectiveness may critically depend on two salient factors: his competence in collecting decision-relevant information, i.e., his "skill", and the alignment of his preferences with those of the top management, i.e., his "loyalty" or bias towards the top. The top management must find the manager's communication believable to use it as a basis for its decisions, and the junior workers must believe that the manager would advocate for their projects and dissuade the headquarters from taking decisions that are detrimental to their interests.

In this paper, we explore how these characteristics of the middle manager—his "loyalty" and "skill"—affect his ability to perform such a dual role, and how loyalty may be managed through the design of communication and decision-making protocols to enhance the manager's effectiveness.

To explore this issue, we first develop a simple static model of a hierarchical organization consisting of a principal, a (middle) manager, and an agent (or "junior worker"). The agent exerts effort to generate a project that may yield certain benefits to all players. The agent obtains a benefit whenever the project is implemented, but the project's value to the manager and the principal is a priori unknown to all parties and is revealed only after its implementation. The manager privately obtains a (noisy) signal on the project's value to the principal and recommends whether to continue or cancel it. Upon receiving the manager's recommendation, the principal decides whether to proceed with the project. The project's value to the principal and the manager may not be perfectly aligned; so, they may have conflicting interests regarding the continuation of the project. The manager's "loyalty" to the top is reflected by the alignment of the project values, and his "skill" by the precision of his signal. In the spirit of the delegation literature, we assume that the manager's report is "cheap talk" and that monetary incentive contracts are infeasible.

The model yields a key insight: the middle manager can effectively perform his dual role only if his loyalty to the top lies within a moderate range, and this range increases with the manager's skill. In other words, the worker would continue to trust a middle manager who is known to be "very loyal to the top" when he is also sufficiently competent; similarly, the headquarters would continue to pay heed to a middle manager who is "too close to the workers" as long as he is smart.

The intuition for this result is as follows. A manager whose value of the project aligns closely with that of the principal is keenly selective in his recommendations as he benefits from the project only when the principal does. In contrast, a manager whose value is more aligned with that of the agent may fail to persuade the top to adopt any project as he is expected to recommend the projects rather indiscriminately. In both cases, the agent correctly anticipates that the ex-ante likelihood of having his project implemented is rather low, and he is discouraged from exerting effort in the first place. But if the manager's value is not too aligned with either the top or the bottom, in equilibrium, he is moderately selective in his recommendation. He recommends implementation often enough so that the agent has adequate incentive to exert effort, but he is also not too lenient in his recommendation and retains credibility with the top.

Furthermore, the manager's skill and loyalty are complements in enhancing his effectiveness. When the manager is more skilled, his signal is more informative, and therefore, the principal's payoff from following the manager's recommendation increases, increasing the manager's credibility with the top. The agent, on the other hand, gets more assured that good projects will not be overlooked, and correctly anticipates a higher likelihood of project implementation.

The static model identifies two situations that may compromise a middle manager's effectiveness. Given his skill level, when the manager is too loyal to the top, he becomes overly selective in his recommendations. Conversely, when he is too loyal to the workers, he becomes undiscerning in his reporting and gets ignored by the principal. In both cases, the manager fails to motivate the agent to take initiative and generate new projects.

The above observation begets the following question: In such situations, what organizational design could the principal adopt to restore the middle manager's effectiveness? We explore this question in the context of a long-term relationship between the principal, the manager, and the agent, where more nuanced communication and project implementation protocols can be sustained. The long-term relationship is modeled as a relational contract between the three players, where they play the aforementioned static game in each period. We focus particularly on the first type of situation identified above, where the middle manager cannot motivate the agent as he is too loyal to the top and consequently too discerning in his report. Such excessive loyalty may reflect, for example, the manager's career ambitions to climb the ranks of the organization's hierarchy. Motivating innovation may require supporting riskier projects with a higher chance of failure. A careerist middle manager may be reluctant to support such projects as a record of past failure may hinder his aspirations for future promotions. Our aim is to understand how an organization where the middle management is excessively loyal to the top can still foster worker initiative and innovation.[1]

Our analysis yields two salient results. First, without loss of generality, we can limit attention to two classes of contracts: (i) A "mediation" contract, where the manager adjusts his behavior by being less stringent in his project recommendation, while the principal always follows the manager's advice. (ii) An "intervention" contract, where the principal sometimes intervenes by directly implementing the project (without considering the manager's report) but otherwise adheres to the manager's recommendation. In other words, incentivizing the agent calls for strategic leniency in project implementation, either mediated through the middle manager or enacted directly by the principal.[2]

In the optimal relational contract, the realized project implementation rate is akin to one that would have been obtained in the static game if the middle manager's loyalty to the top were somewhat more moderate. That is, in the optimal mediation contract, relational incentives induce the manager to behave as one who is (in expectation) less loyal to the top than he really is. In contrast, under the optimal intervention contract, relational incentives induce the principal to behave in a way that, in expectation, offsets the manager's excessive loyalty.

It is worth noting that the two contracts we highlight mirror real-world management strategies that seek to foster innovation. In line with mediation contracts, firms often promote a culture tolerant of failure to encourage idea generation. Farson and Keyes (2002) document how CEOs at GE, Coca-Cola, and Lockheed created "risk-friendly workplace" where the managers were encouraged to pursue ideas that may have a high risk of failing. Similarly, House and Price (2009) describe how HP division managers were informally expected to allocate part of their R&D budgets to high-risk exploratory projects.[3] Direct interventions by leadership are also commonplace. Thomke (2002) details how Bank of America's headquarters allocated funds to selected branches to develop new products and services, explicitly encouraging high failure rates to spur experimentation. In a study of a fast-growing social-media marketing firm, Turco (2016) highlights "Hack Night" events where employees pitched ideas directly to top executives, often receiving immediate support to pursue them. The corporate policy of granting discretionary time to employees to pursue any project they

[1] The second type of situation—where the manager is too indiscriminate in his reporting—is reminiscent of the incentive problem studied in the literature on delegation in organizations (Gibbons et al. 2011; Li et al. 2017). We examine this case later in Section 7.1 within the framework of our model, specifically in the context of a hierarchical organizational structure.

[2] The role of strategic leniency in incentive provision has been recognized in other contexts as well; see, e.g., Levitt and Snyder (1997) and Fairburn and Malcomson (2001).

[3] See Manso (2011) for a theoretical model of incentives for innovation that highlights the optimality of tolerance (and even reward) for failure.

perceive as beneficial to the organization, such as Google's "20% rule" used in the early 2000s, also resembles the structure of intervention contracts.

Second, we show that the optimality of each of these two classes of contracts depends on the characteristics of the middle manager. In this setup, a key parameter is the value a player obtains from a successful project relative to his/her outside option. In particular, if the ratio of project value to outside option for the manager is sufficiently small, the mediation contract is not feasible. In contrast, if this ratio is large enough, the mediation contract becomes (weakly) optimal. Otherwise, either mediation or intervention (or both) may be optimal depending on the manager's loyalty and skill. When the manager's skill and loyalty are both sufficiently high, only intervention is optimal; when both are sufficiently low, only mediation is optimal; and finally, if the manager is too loyal but incompetent, then no relational contract (that induces the agent to exert effort in project generation) can be sustained.

Our results offer two important insights on how personnel characteristics may affect organizational design. First, note that in the optimal contract, the characteristics of the middle manager—i.e., his loyalty and skill—shape the nature of communication and decision-making within the organization.

Under mediation, the principal always follows the manager's recommendation, and the manager slants his recommendation in the agent's favor to sharpen incentives. In other words, the principal effectively delegates the task of incentivizing the agent to the middle manager. But under intervention, the principal takes a direct role in incentivizing the junior workers by occasionally deciding on their projects without considering the manager's recommendation. Thus, our result indicates that the "managerial style"—e.g., the pattern of decision-making—that one may observe in an organization may stem from the personal characteristics of the middle management rather than that of the CEO herself. This finding is in sharp contrast to the literature on managerial fixed effects that explores correlations between the corporate policies a CEO adopts and her own characteristics (Bertrand and Schoar, 2003; Dessein and Santos, 2021; Lo et al., 2022).

Second, our result implies that a manager can effectively mediate between the workers and top management only if he exhibits sufficient loyalty to the organization, as reflected by his project value to outside option ratio. In other words, only the middle managers who place a high enough value on working for the organization may be induced to adjust their behavior to motivate the frontline workers. Furthermore, in the optimal contract, the manager's loyalty to the organization interacts with his loyalty to the top management. A middle manager perceived as too loyal to the top can still mediate effectively—provided he is also keenly loyal to the organization itself.

Conventional wisdom regards employee loyalty to leadership as a virtue, but in hierarchical organizations, it can be a double-edged sword. In particular, excessive loyalty of middle management to the top can weaken frontline workers' incentives. We show that loyalty can be managed and tempered through well-designed communication and decision-making protocols. The optimal protocols that we highlight—mediation and intervention—resonate with two contrasting perspectives on middle management: as either a catalyst for innovative ideas or an administrative hurdle that may be avoided to expedite innovation. Our analysis indicates that, depending on the manager's characteristics, both views can be valid. However, as we further elaborate below, the optimal protocols are more nuanced than what these broad views would suggest, and the implementation of each protocol requires leveraging distinct relationships within the organization.

Related literature: The importance of middle managers for firm performance has been long recognized in the management literature (Dalton, 1959; Chandler, 1977; Burgelman, 1983a, 1983b, 1991, 1994; Mintzberg, 1989; Osterman, 2009; Jaser, 2021) and, more recently, in economics (Lazear et al. 2015; Bianco and Giorcelli, 2022; Fenizia, 2022; Friebel et al. 2022; Friebel and Raith, 2022; Metcalfe et al. 2023; also see Roberts and Shaw, 2022, for a survey of this literature). Recent empirical studies have shown that middle managers can reduce turnover with better interpersonal skills (Hoffman and Tadelis, 2021) and improve productivity via attention to worker-to-task matches (Adhvaryu et al. 2022). Additionally, the managers' tenure, cognitive skills, and autonomy positively affect their rate of learning on the job and enhance production efficiency (Adhvaryu et al. 2023).

However, the critical role of the middle managers in linking an organization's top management with its frontline workers has not received much attention in the literature. In particular, the literature offers little insight into the personal characteristics that make the middle managers succeed in such a role that demands "dual loyalty." Our paper attempts to fill this gap by exploring how the middle managers' effectiveness is driven by the interplay among their loyalty within and to the organization and their managerial skills. Furthermore, we highlight how such characteristics of the middle managers influence the communication and decision-making processes within the organization.

Following the seminal work by Hirschman (1970), a large literature has explored the relationship between employee loyalty to the organization and its performance (see, e.g., a survey by Guillon and Cezanne, 2014). In the context of political economy, several authors have also explored a trade-off a leader may face between hiring a more skilled or more loyal agent (Egorov and Sonin, 2011; Zakharov, 2016; Dessein and Garicano, 2025). However, little has been studied about the relationship between organizational performance and the distribution of loyalty within its hierarchy, where higher employee loyalty to one tier (e.g., the top) may necessarily imply lower loyalty towards another (i.e., the bottom). We contribute to this literature by showing how such distribution of loyalty within the organization affects its performance, how it can be managed to enhance organizational performance, and how loyalty within the organization interacts with loyalty towards the organization.

Our paper relates to several strands of literature in organizational and personnel economics. Various authors have highlighted how incentives to innovate are stronger when the agents are more assured of support from their supervisors (Rotemberg and Saloner, 1993, 2000; Aghion et al. 2013; Nguyen, 2023). In this strand of papers, our setup is closest to the two papers by Rotemberg and Saloner. Rotemberg and Saloner (1993) consider a model of hierarchical organization (with shareholders, CEO, and a manager) to highlight the value of a CEO who is altruistic towards the manager who must invest in developing new projects. However, in their model, the top rung of the organization, i.e., the shareholders, does not directly participate in the production process. Hence, the CEO's job does not involve linking the top and the bottom rungs of the organization and does not require maintaining dual credibility. Rotemberg and Saloner (2000) show that a CEO with a known bias towards certain types of projects may be more effective in motivating the agents, as projects that align with the CEO's bias have a greater chance of getting implemented. In their model, all rungs of the organization participate in the production process. In contrast to our paper, they abstract away from the dual role of middle management and highlight the value of "unbiased" middle managers who may allocate funds to projects that may not align with the CEO's vision but nonetheless could become profitable for the firm.

There is a vast literature on incentives in hierarchical organizations that have primarily focused on issues of monitoring and delegation (for a survey, see Mookherjee, 2013). On the topic of monitoring, this literature explores the role of middle managers as supervisors to the employees, where the supervisory task itself may be subject to a moral hazard problem. Following a seminal contribution by Tirole (1986), several papers have explored the implications of such contracting frictions on the provision of incentives within the organization (Kofman and Lawarree, 1993; Olsen and Torsvik, 1998; Burguet and Che, 2004). Bearing resemblance to our modeling framework, a more recent contribution by Troya-Martinez and Wren-Lewis (2023) models the firm-manager-supplier relationship as a relational contract where the manager and the supplier may collude on the price and quality of the product. Allowing for monetary transfers, they argue that side payments or "kickbacks" from the agent to the manager can facilitate incentive provision by decreasing the manager's temptation to renege on her promise. The literature on delegation, on the other hand, explores the trade-off between loss of incentives/coordination and loss of control as the firm decides on whether to delegate decision rights to a biased but potentially more informed manager (Melumad et al. 1995; Aghion and Tirole, 1997; Baker et al. 1999; Athey and Roberts, 2001; Alonso and Matouschek, 2007; Alonso et al., 2008; Dessein, 2002; Rantakari, 2008, 2012; Li et al., 2017; Lipnowski and Ramos, 2020).

Our paper contrasts with and contributes to this literature by highlighting how the effectiveness of a middle manager requires careful management of his dual loyalty, but we abstract away from the issue of monitoring the agents' effort and the control-coordination trade-off with decentralized decision-making.

This article also contributes to the literature on strategic information revelation in relational contracts. Studying a varied range of environments, several authors have shown how deliberate manipulation of information can make relational contracts more efficient (Fuchs, 2007; Kvaloy and Olsen, 2009; Fong and Li, 2017; Orlov, 2022; Wong, 2023). When messages are cheap talk, repeated interaction may also discipline the sender to communicate truthfully (Kolotilin and Li, 2021; Kuvalekar, et al. 2022). However, these authors study principal-agent relationships rather than communication in organizational hierarchies, and we explore how strategic communication between the middle manager and the top management affects the incentive for the bottom worker.

In the aforementioned literature, Troya-Martinez and Wren-Lewis (2023) and Fong and Li (2017) deserve particular mention as, to the best of our knowledge, these are the only two papers that consider relational contracts in an organizational hierarchy. However, in both papers, at least one player does not have any dynamic incentives as they do not actively participate in the relationship. In Troya-Martinez and Wren-Lewis (2023), the principal chooses some parameters of the contract at the beginning of the game and lets the middle manager and the agent engage in a relational contract, whereas in Fong and Li (2017), the middle manager is modeled as a non-strategic "supervisor" who simply relays information as per a pre-specified reporting rule. In contrast, we consider a setup where the players in all three rungs of the organization actively participate in a relational contract, and explore how the interplay among their dynamic incentive constraints shapes the effectiveness of the middle manager and influences the communication and decision-making within the organization.

The remainder of the article is organized as follows: In Section 2, we present a static model that serves as a baseline for our analysis, and it is analyzed in Section 3. Section 4 embeds the baseline model in an infinitely repeated game and Section 5 explores the feasibility of mediation and intervention contacts. The optimal relational contract is characterized in Section 6. A final section, Section 7, presents a few extensions of our analysis and draws a conclusion. All proofs are given in the Appendix.

2 A baseline model

In this section, we develop a simple model of a hierarchical organization where a principal $\mathcal{P}$ (or "firm") hires a (middle) manager $\mathcal{M}$ and an agent $\mathcal{A}$ (or "junior worker") who may work together on a project. The model illustrates how the characteristics of the middle manager in terms of his "loyalty" and "skill" affect his credibility with his supervisor and the work incentives of his subordinates, and it will later serve as the "stage game" in a more general model with repeated interaction.

We elaborate on the model by describing its three key components: *Technology, information structure and reporting,* and *payoffs.*

Technology: The agent (publicly) exerts costly effort $e \in \{0, 1\}$ to generate a project that may yield private value to all three players. The cost of effort is $c(e)$, where $c(0) = 0$ and $c(1) = c$. A project is generated if and only if $e = 1$. Once a project is generated, the manager evaluates its potential value and reports to the principal on whether she should implement it (we will elaborate on the manager's reporting decision shortly). If the project is implemented, the agent obtains a value of b. The project's value to the principal, $V_{\mathcal{P}} \in \{0, Y\}$, is stochastic where $V_{\mathcal{P}} = 0$ or Y with equal probability. However, the project's value to the manager, $V_{\mathcal{M}} \in \{0, y\}$, depends on the project's type $\omega \in \{C, D\}$. If $\omega = C$, the manager's value is "congruent" with that of the principal: $V_{\mathcal{M}} = y$

if and only if $V_{\mathcal{P}} = Y$. If, instead, $\omega = D$, their values "diverge" as $V_{\mathcal{M}} = y$ regardless of $V_{\mathcal{P}}$. The state ω is a priori unknown to all parties who hold a common prior $\Pr(\omega = C) = \alpha$.

Note that for high α the manager is more likely to benefit from the project only if the principal also benefits from it. In that case, the manager would be more discerning in his recommendation regarding the project's implementation. In contrast, for low α, the manager is more likely to always benefit from the project's implementation—as is the agent—and would be indiscriminate with his recommendation to implement the project. Thus, α reflects how aligned the manager's preference is with that of the principal vis-a-vis the agent, and one may interpret α as the "loyalty" of the manager towards the principal.

The production environment stated above may be mapped into a wide range of real-world scenarios. Even though the agent is in charge of generating the project, the middle manager's personal characteristics may influence the type of project that the agent can generate as the manager can facilitate the process by ensuring task coordination, supplying complementary inputs and guidance. A manager may manipulate the process in a way that ensures that the project is always profitable to him regardless of its value to the principal. Now, α simply reflects the likelihood that the manager refrains from such influence activities. Alternatively, one may consider a setup where the project can be evaluated on multiple dimensions. Suppose that the project has two aspects; though the middle manager may care about both, the principal only cares about the first one. In particular, if the second aspect is above a certain threshold, the manager wants to proceed with the project regardless of the other aspect of the project. But if the second aspect is below the threshold, then whether he wants to proceed or not depends on the first aspect. The manager's loyalty, α, may reflect the probability that a project's second aspect is below the threshold, and hence, the manager now cares about the first aspect only, as does the principal. Thus a more loyal manager is one with a higher threshold in the second aspect of a project, i.e., a manager who is unlikely to assign any special importance to the second aspect.[4]

Information structure and reporting: Once a project is generated, the manager observes its type ω and a signal $s \in \{0, 1\}$ on the principal's payoff from the project, where

$$\Pr(s = 1 \mid V_{\mathcal{P}} = Y) = \sigma \text{ and } \Pr(s = 1 \mid V_{\mathcal{P}} = 0) = 0.$$

We interpret $\sigma \in (0, 1)$ as the "skill" of the manager as it captures the manager's ability to identify good projects for the principal. Though the signal is noisy, $s = 1$ indicates a high-value project for the principal with certainty. Upon observing the signal, the manager sends a report $r \in \{0, 1\}$ to the principal. The report is "cheap talk" that need not align with the manager's observed signal. Observing the manager's report, the principal decides whether to implement the project or cancel

[4]As a concrete example, consider a firm where innovation is crucial. The firm employs researchers who have to take the initiative to develop new projects with very uncertain outcomes. Suppose the principal cares about the short-term profitability of the project. The middle manager, on the other hand, may care about the profitability and the value of scientific discovery. If the scientific discovery is novel enough, the middle manager wants the project to proceed (regardless of the profitability). If the scientific discovery is not quite significant, then the middle manager will look at the project's profitability and prefer to continue only if it is profitable to do so.

it. We denote the principal's implementation decision by $\iota \in \{0,1\}$, where $\iota = 1$ if the project is implemented and 0 otherwise.

The table below summarizes the project values for the principal and the manager given the project type and the manager's signal.

	$s=1$ (prob. $\frac{1}{2}\sigma$)	$s=0$ (prob. $1-\frac{1}{2}\sigma$)
$\omega = D$ (prob. α)	(Y,y)	(Y,y) or $(0,y)$
$\omega = C$ (prob. $1-\alpha$)	(Y,y)	(Y,y) or $(0,0)$

Table 1: Principal and manager's project values, $(V_{\mathcal{P}}, V_{\mathcal{M}})$, given the state and signal (ω, s).

Payoffs: All players are risk neutral. If any one of the three players decides not to participate, all players receive their outside options. We assume that the outside option for the agent is 0, whereas the outside options of the manager and the principal are v_0 and π_0, respectively. These are also the payoffs of the three players if the agent withholds effort, i.e., $e = 0$, and no project is generated.

Thus, if all players participate, given the project type ω, manager's signal s, and his report r, the payoff of the agent, manager, and the principal are:

$$\hat{u} := \begin{cases} b-c & \textit{if } e=1 \textit{ and } P \textit{ implements the project} \\ -c & \textit{if } e=1 \textit{ and } P \textit{ does not implement the project} \\ 0 & \textit{otherwise} \end{cases},$$

$$\hat{v} := \begin{cases} \mathbb{E}\left[V_{\mathcal{M}} \mid s,\omega\right] & \textit{if } e=1 \textit{ and } P \textit{ implements the project} \\ v_0 & \textit{otherwise} \end{cases},$$

and

$$\hat{\pi} := \begin{cases} \mathbb{E}\left[V_{\mathcal{P}} \mid r\right] & \textit{if } e=1 \textit{ and } P \textit{ implements the project} \\ \pi_0 & \textit{otherwise} \end{cases}.$$

respectively.

We impose the following restrictions on the parameters in order to focus on a more interesting modeling environment.

Assumption 1. *(i)* $\frac{1}{2}Y < \pi_0 < Y$; *(ii)* $\frac{1}{2}y < v_0 < y$; and *(iii)* $b > 2c$.

Assumption 1 (i) implies that it is optimal for the principal to implement the project when it is known to be of high value, but it is not worth implementing solely based on her prior belief about the project's valuation. In a similar spirit, Assumption 1 (ii) implies that the middle manager would always prefer to have the project implemented if he is sure to have a high value but would prefer to have it canceled if he only knows that the state is C, and hence, both high and low values are equally likely. Finally, the parametric restriction in (iii) is imposed to streamline the analysis by ruling out certain boundary solutions.

TIMELINE: The timing of the game is summarized below.

- *Stage* 1. The principal, the manager and the agent decide whether to engage in the production process. If at least one of the three players refuses to participate, the game ends. Otherwise, the game moves to Stage 2.
- *Stage* 2. The agent chooses his effort level e. If $e = 0$, the game ends. Otherwise, the game moves to stage 3.
- *Stage* 3. The manager observes the project type ω and signal s.
- *Stage* 4. The manager sends a report r to the principal.
- *Stage* 5. The principal decides whether to implement the project.
- *Stage* 6. All parties receive their payoffs, and the game ends.

STRATEGIES AND EQUILIBRIUM: The strategies of all three players have two components each. For each player, the first component of their respective strategies is a decision on whether to participate in the production process. The second component of the strategies differs across the players. For the agent, it specifies an *effort choice* $e \in \{0, 1\}$. For the manager, it specifies a *reporting policy* that maps his information on the state $\omega \in \{C, D\}$ and his signal $s \in \{0, 1\}$ to his report $r \in \{0, 1\}$. Finally, for the principal, it specifies an *implementation policy* that maps the manager's report to a decision on whether to implement or cancel the project. We use perfect Bayesian Equilibrium as our equilibrium concept.

3 Analyzing the baseline model

This section characterizes the equilibrium in our baseline model, and, in the process, highlights the key economic effects that are in play. In particular, it illustrates how the characteristics of the middle manager—his "loyalty" and "skill"—affect his ability to achieve the dual goals of maintaining credibility with the top management and encouraging work initiative among his subordinates as he strategically reports his information to the principal.

We begin our analysis with the following observation. From the principal's perspective, the ideal outcome is one where the agent exerts effort, and the project is implemented if and only if the associated signal is favorable (i.e., $s = 1$). However, two frictions in our model prevent the principal from achieving this outcome. First, the principal cannot directly observe the signal on project quality and must rely on the manager's report when deciding on project implementation. The manager, in turn, may manipulate the report to influence the principal's decision. Second, the decision rule that results from this communication may lead to an ex-ante likelihood of project implementation that is insufficient to incentivize the agent to generate a project in the first place.

In any equilibrium of this model, these two frictions compromise the principal's payoff. In particular, as the manager's report is "cheap talk", there always exists a "babbling equilibrium" where the manager's report is completely uninformative (e.g., he sends the same report regardless of his information) and, in response, the principal always ignores the manager's report. As no project gets implemented, the agent does not generate any project in the first place, and all players take their respective outside options. In what follows, we focus on the equilibrium that, given the middle manager's characteristics, yields the highest payoff to the principal.

In any informative equilibrium (i.e., one where the manager's recommendation influences the principal's implementation decision), the manager would always recommend implementation if $\omega = D$ or $(\omega, s) = (C, 1)$, as in both cases his value of the project (y) exceeds his outside option (v_0). Thus, the only informative communication that may be sustained in equilibrium is one where the principal implements the project only if the manager recommends implementation by sending $r = 1$ (say) and the manager chooses to send $r = 1$ except when $(\omega, s) = (C, 0)$.

Such a communication protocol induces the agent to exert effort if and only if $\Pr(\iota = 1 \mid e = 1)\, b = \Pr(r = 1 \mid e = 1)\, b \geq c$, i.e.,

$$1 - \alpha\left(1 - \frac{1}{2}\sigma\right) \geq \frac{c}{b}. \qquad (IC_{\mathcal{A}})$$

(Trivially, $\Pr(\iota = 1 \mid e = 0) = 0$ as no project is generated in the first place.) One may readily check that the manager cannot gain from deviating from the above reporting strategy. As mentioned above, he obtains $y > v_0$ in all states where he reports $r = 1$, and when $(\omega, s) = (C, 0)$, his expected payoff from the project is $\frac{1-\sigma}{2-\sigma} y$, which, by Assumption 1 (ii), is smaller than v_0, the payoff he gets by reporting $r = 0$ and having the project canceled.

Finally, consider the principal's incentive to follow the manager's recommendation. It is optimal for her to follow the manager's recommendation if and only if

$$\Pr(V_{\mathcal{P}} = Y \mid r = 1)\, Y \geq \pi_0,$$

i.e.,

$$\frac{\frac{1}{2}\sigma + \frac{1}{2}(1-\sigma)(1-\alpha)}{1 - \alpha\left(1 - \frac{1}{2}\sigma\right)} Y \geq \pi_0, \qquad (IC_{\mathcal{P}})$$

and $\Pr(V_{\mathcal{P}} = Y \mid r = 0)\, Y < \pi_0$, i.e.,

$$\frac{1-\sigma}{2-\sigma} Y < \pi_0. \qquad (IC'_{\mathcal{P}})$$

Note that $(IC'_{\mathcal{P}})$ is always satisfied under Assumption 1 (i). Thus, the aforementioned strategies of the principal and the manager, along with $e = 1$, can be supported in equilibrium if and only if ($IC_{\mathcal{A}}$) and ($IC_{\mathcal{P}}$) are satisfied. The proposition below characterizes this equilibrium and the conditions under which it can be sustained.

Proposition 1. *The equilibrium that yields the largest payoff to the principal is characterized as follows. There exist $\sigma_0 \in (0, 1)$ such that:*

(*i*) *If $\sigma < \sigma_0$, no project is generated, and the players obtain their outside options.*

(*ii*) *If $\sigma \geq \sigma_0$, there exist two cutoffs, α_0 and α_1, where $0 < \alpha_0 < \alpha_1 \leq 1$, such that:*

- (*a*) *If $\alpha \in [\alpha_0, \alpha_1]$, the agent generates a project (i.e., $e = 1$), the manager reports $r = 0$ when $(\omega, s) = (C, 0)$ and reports $r = 1$ otherwise, and the principal implements the project if and only if $r = 1$.*
- (*b*) *If $\alpha \notin [\alpha_0, \alpha_1]$, no project is generated, and all players obtain their outside options.*

The cutoff α_0 is decreasing in σ whereas α_1 is increasing in σ, with $\alpha_1 = \alpha_0$ if $\sigma = \sigma_0$ and $\alpha_1 = 1$ if $\sigma \geq \sigma_1$ for some $\sigma_1 \in (\sigma_0, 1)$.

Proposition 1 shows how the agent's incentives and communication between the manager and the principal are affected by the manager's characteristics, namely, his skill and loyalty to the principal. When the precision of the manager's signal is low ($\sigma < \sigma_0$), the principal is always better off taking her outside option, and hence, no project is generated. But when the signal is sufficiently precise ($\sigma \geq \sigma_0$), production may take place provided that the manager's loyalty to the principal is in an intermediate range.

Figure 1 illustrates the equilibrium characteristics as stated in the above proposition. There exists an equilibrium where the agent exerts effort if and only if the parameters (α, σ) are in region A. For example, when the manager's skill $\sigma = \sigma'$, in this equilibrium, the agent exerts effort and the principal follows the manager's recommendation provided that the manager's loyalty to the principal, α, is in $[\alpha_0', \alpha_1']$. In all other regions, no equilibrium exists where the agent exerts effort to generate a project. In region B, ($IC_{\mathcal{A}}$) is violated, in region D, ($IC_{\mathcal{P}}$) is violated, and in region C, the signal precision is so low that the principal prefers to take her outside option rather than engaging in production.

To see the intuition for this result, recall the dual goal of the manager's reporting policy: the manager needs to recommend the agent's project sufficiently often so as to incentivize the agent to exert effort, but the manager may not be too indiscriminate in his recommendation lest he loses credibility with the principal. In other words, for the manager's reporting policy to be effective in meeting these two goals, both ($IC_{\mathcal{A}}$) and ($IC_{\mathcal{P}}$) must hold. Now, for a given "skill" level (σ), the more loyal the manager is, i.e., the higher is α, the more likely it is that the project is of value to him *only when* it is also of value to the principal. Consequently, he becomes more discerning in his recommendation; but as it lowers the likelihood that the project would be implemented, he fails to incentivize the agent. On the other hand, when the manager's loyalty to the principal is sufficiently

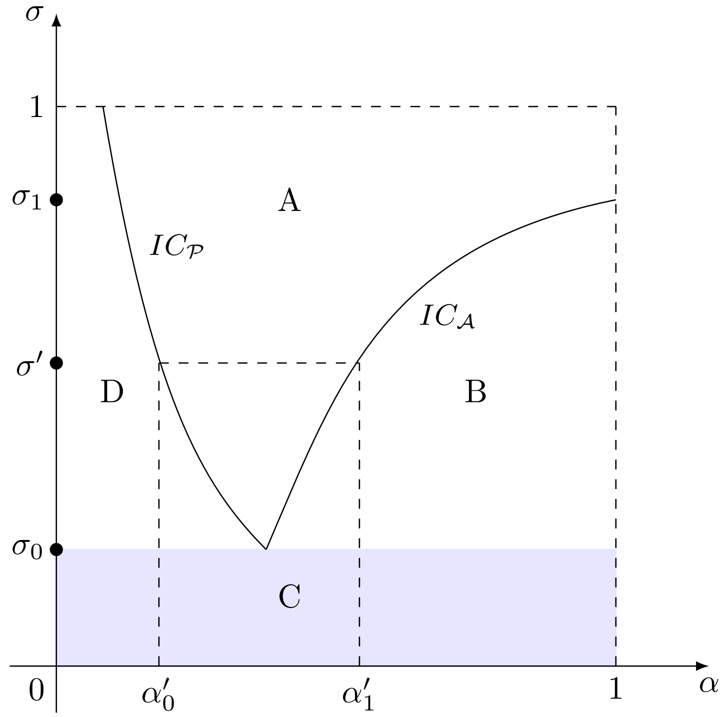

Figure 1: Equilibrium characterization in the baseline model.

low, he is too indiscriminate in his recommendation, and the principal disregards his report. Thus, the manager can attain the dual goal of his reporting policy in equilibrium only if he is truly in the "middle"—not too loyal to the top and also not too close to the bottom.

Proposition 1 also highlights a complementarity between the manager's skill and loyalty: the range of loyalty, $[\alpha_0, \alpha_1]$, that can support production in equilibrium, increases in the manager's skill, σ. In other words, both ($IC_{\mathcal{A}}$) and ($IC_{\mathcal{P}}$) get relaxed with higher σ. As the manager's signal gets more precise, the principal's expected payoff from following the manager's recommendation (to implement the project) increases. The agent, on the other hand, gets more assured that his effort will lead to project implementation as the manager is more likely to detect a good project when one is generated.

4 Communication and decision-making under relational incentives

The above analysis of the baseline model highlights two types of distortions that compromise the principal's payoff: first, in region A the project is implemented too often relative to what the principal would prefer; second, in regions B and D the project is not implemented at all, in fact it is not generated in the first place as the agent lacks incentives for exerting effort. The distortion in regions A and D arises because the manager is not loyal enough to the principal and recommends implementation too often (in region D in particular, the manager's recommendation completely loses its credibility). In contrast, the distortion in region B stems from the fact that the manager is too loyal to the principal and does not recommend implementation often enough. We explore how such inefficiencies may be ameliorated in a long-term employment relationship where the parties may rely on relational incentives to influence the likelihood of project implementation.

The remedy for the first type of problem (regions A and D) calls for discouraging the manager from recommending projects that are beneficial to him but detrimental to the principal. Such an incentive problem is reminiscent of the seminal work by Aghion and Tirole (1997) on delegation and the literature that followed.[5] In contrast, the remedy for the latter type of problem (region B) calls for incentivizing the manager and the principal to be more lenient and allow the implementation of projects that they would rather have canceled. This is a novel problem that has hitherto received little attention in the literature, and we focus our analysis on this case.

4.1 A model of relational contract

We consider an infinitely repeated game where the principal, manager and agent enter in a long-term employment relationship. Time is assumed to be discrete and denoted as $t \in \{1, 2, ..., \infty\}$. All players have a common discount factor $\delta \in (0, 1)$. In each period, the players play a stage game that is identical to the game described in our baseline model, except for the following modifications.

We assume that the project type is i.i.d across periods. In order to allow for more nuanced reporting and implementation policies, we introduce two randomization devices that the manager and the principal may use in their respective reporting and implementation decisions. After the agent exerts effort, all players observe the realization x_t of a public randomization device. In addition, the manager and the agent also observe the realization z_t of a randomization device that is not observed by the principal. Both x_t and z_t are independently and uniformly distributed on $[0, 1]$.

The manager, given his information on the state and signal (ω_t, s_t) and the realization of the randomization devices, (x_t, z_t), sends a report $r_t \in \{0, 1\}$ to the principal. The report is observed by all the players. Given the manager's report and the realization of the public randomization device x_t, the principal makes her decision on implementation, $\iota_t \in \{0, 1\}$ ($\iota_t = 1$ if the project is implemented and 0 otherwise).

The timeline of the stage game is illustrated in Figure 2. We denote the participation decision of player $i \in \{\mathcal{A}, \mathcal{M}, \mathcal{P}\}$ as $d_t^i \in \{0, 1\}$, where $d_t^i = 1$ if and only if player i decides to participate in period t. Furthermore, we assume that the project quality—that is, the principal's realized payoff from an implemented project—is revealed at the end of the stage game for each period.

Histories: The particular information structure of the game leads us to define three different histories. First, let $h_t^{\mathcal{P}} = \{d_t^{\mathcal{A}}, d_t^{\mathcal{M}}, d_t^{\mathcal{P}}, e_t, x_t, r_t, \iota_t, V_{\mathcal{P},t}\}$ be the public history of period t where $V_{\mathcal{P},t}$ denotes the principal's realized payoff in period t (set $V_{\mathcal{P},t} = \emptyset$ if $\iota_t = 0$), $h^{\mathcal{P},t} = (h_1^{\mathcal{P}}, h_2^{\mathcal{P}}, ..., h_{t-1}^{\mathcal{P}})$ be the public history up to date t, and $H^{\mathcal{P},t}$ be the set of all possible date t histories (set $H^{\mathcal{P},1} = \emptyset$). Note that the set of the principal's date t histories is $H^{\mathcal{P},t}$. Next, to obtain the sets of the agent's

[5] See, e.g., Gibbons et al. (2012) for a survey of this literature.

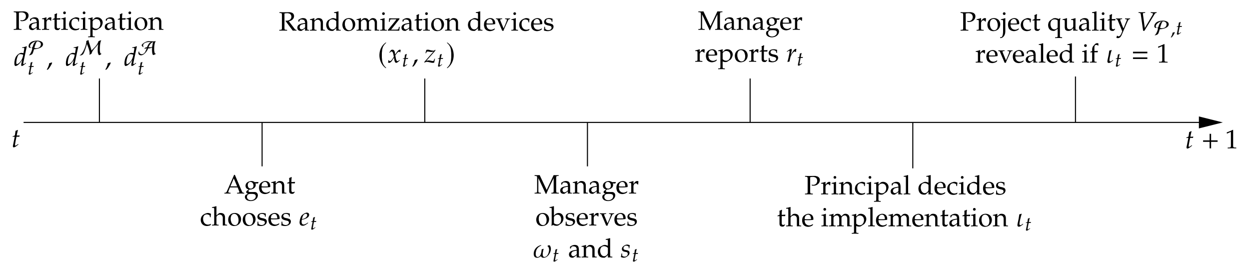

Figure 2: The timeline of the stage game in the repeated version.

and the manager's date t histories, we combine their private information with the public history. Let $h_t^{\mathcal{A}} := h_t^{\mathcal{P}} \cup \{z_t\}$ and $h_t^{\mathcal{M}} := h_t^{\mathcal{P}} \cup \{z_t, \omega_t, s_t\}$ be the private history of period t observed by the agent and the manager, respectively. Denote their respective private histories up to date t as $h^{\mathcal{A},t} = (h_1^{\mathcal{A}}, ..., h_{t-1}^{\mathcal{A}})$ and $h^{\mathcal{M},t} = (h_1^{\mathcal{M}}, ..., h_{t-1}^{\mathcal{M}})$, and let $H^{\mathcal{A},t}$ and $H^{\mathcal{M},t}$ be the set of all possible private histories up to date t for the agent and the manager.

Strategies: A strategy of the agent specifies a sequence of functions $\sigma^{\mathcal{A}} := \{(\sigma_{t,1}^{\mathcal{A}}, \sigma_{t,2}^{\mathcal{A}})\}_{t=1}^{\infty}$. For date t, the function $\sigma_{t,1}^{\mathcal{A}} : H^{\mathcal{A},t} \to \{0,1\}$ specifies his participation decision, and the function $\sigma_{t,2}^{\mathcal{A}} : H^{\mathcal{A},t} \to \{0,1\}$ specifies the agent's effort choice.

Similarly, a strategy of the manager stipulates a sequence of functions $\sigma^{\mathcal{M}} := \{(\sigma_{t,1}^{\mathcal{M}}, \sigma_{t,2}^{\mathcal{M}})\}_{t=1}^{\infty}$. The function $\sigma_{t,1}^{\mathcal{M}} : H^{\mathcal{M},t} \to \{0,1\}$ specifies the manager's participation decision for date t based on his history. The function $\sigma_{t,2}^{\mathcal{M}} : H^{\mathcal{M},t} \times \{0,1\} \times [0,1]^2 \times \{C, D\} \times \{0,1\} \to \{0,1\}$ specifies the manager's reporting decision in date t given his history, the agent's effort e_t, the realization of the two randomizing devices (x_t, z_t), and his observation on the project's state and the signal (ω_t, s_t).

Finally, a strategy of the principal stipulates a sequence of functions $\sigma^{\mathcal{P}} := \{(\sigma_{t,1}^{\mathcal{P}}, \sigma_{t,2}^{\mathcal{P}})\}_{t=1}^{\infty}$. The function $\sigma_{t,1}^{\mathcal{P}} : H^{\mathcal{P},t} \to \{0,1\}$ specifies the principal's participation decision in period t given her history. The function $\sigma_{t,2}^{\mathcal{P}} : H^{\mathcal{P},t} \times \{0,1\} \times [0,1] \times \{0,1\} \to \{0,1\}$ specifies her implementation decision for date t given her history, the agent's effort e_t, the realization of the public randomizing device x_t, and the manager's recommendation r_t.

Payoffs: With a slight abuse of notation, we denote a profile of strategies $(\sigma^{\mathcal{A}}, \sigma^{\mathcal{M}}, \sigma^{\mathcal{P}})$ as $\boldsymbol{\sigma}$. Given a strategy profile $\boldsymbol{\sigma}$ and a history up to date t, $h^t \in H^{\mathcal{A},t} \cup H^{\mathcal{M},t} \cup H^{\mathcal{P},t}$, the expected payoffs of the players from the continuation game starting in period t are defined as follows. For the agent:

$$u_t(h^t, \boldsymbol{\sigma}) = (1-\delta)\mathbb{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} d_\tau \hat{u}_\tau \;\middle|\; h^t, \boldsymbol{\sigma}\right],$$

for the manager:

$$v_t(h^t, \boldsymbol{\sigma}) = (1-\delta)\mathbb{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(d_\tau \hat{v}_\tau + (1-d_\tau) v_0\right) \;\middle|\; h^t, \boldsymbol{\sigma}\right],$$

and for the principal:

$$\pi_t(h^t, \boldsymbol{\sigma}) = (1-\delta)\mathbb{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t}\left(d_\tau \hat{\pi}_\tau + (1-d_\tau)\pi_0\right) \,\middle|\, h^t, \boldsymbol{\sigma}\right],$$

where $d_\tau := d_t^{\mathcal{A}} d_t^{\mathcal{M}} d_t^{\mathcal{P}} = 1$ if all players decide to participate in production in period τ, and $d_\tau = 0$ otherwise. When no confusion arises, we omit the strategy profile $\boldsymbol{\sigma}$ in the payoff expressions. Thus, the tuple (u_1, v_1, π_1) represents the players' payoffs at the beginning of the repeated game.

We refer to $u_t(h^{\mathcal{A},t}), v_t(h^{\mathcal{M},t}), \pi_t(h^{\mathcal{P},t})$—i.e., the agent, the manager, and the principal's expected payoffs in the continuation game conditional on the history that they individually observe—as the players' respective continuation payoffs at the beginning of period t.

Equilibrium: A pure-strategy perfect Bayesian equilibrium (referred to below simply as an "equilibrium") consists of a strategy profile $\boldsymbol{\sigma} = (\sigma^{\mathcal{A}}, \sigma^{\mathcal{M}}, \sigma^{\mathcal{P}})$ and the players' posterior beliefs $\boldsymbol{\mu} = (\mu^{\mathcal{A}}, \mu^{\mathcal{M}}, \mu^{\mathcal{P}})$ that stipulate each player i's belief about the others' private histories given his/her own private history $h^{i,t}$, $(i \in \{\mathcal{A}, \mathcal{M}, \mathcal{P}\})$. In our model, the manager's private history contains more information than the agent's, and the agent's private history contains more information than that of the principal's (which is the public history of the game). Thus, the agent and the manager's beliefs about the history observed by the principal (i.e., $\mu^{\mathcal{A}}(h^{\mathcal{P},t} \mid h^{\mathcal{A},t})$, and $\mu^{\mathcal{M}}(h^{\mathcal{P},t} \mid h^{\mathcal{M},t})$) are necessarily degenerate, and so is the manager's belief about the agent's private history (i.e., $\mu^{\mathcal{M}}(h^{\mathcal{A},t} \mid h^{\mathcal{M},t})$). The posterior beliefs $\mu^{\mathcal{A}}(h^{\mathcal{M},t} \mid h^{\mathcal{A},t})$, $\mu^{\mathcal{P}}(h^{\mathcal{A},t} \mid h^{\mathcal{P},t})$, and $\mu^{\mathcal{P}}(h^{\mathcal{M},t} \mid h^{\mathcal{P},t})$ stipulate how the agent and the principal assign (non-degenerate) probabilities to the private histories of the other players, conditional on their own observed histories.

The strategies and posterior beliefs constitute an equilibrium if they satisfy the following two conditions:

(i) Best Response: For each player $i \in \{\mathcal{A}, \mathcal{M}, \mathcal{P}\}$, σ^i is a best response to the other players' strategies after every private history $h^{i,t}$, given the beliefs $\boldsymbol{\mu}$.

(ii) Belief Consistency: The beliefs $\boldsymbol{\mu}$ are consistent with the strategy profile $\boldsymbol{\sigma}$ and are updated using Bayes' rule whenever possible.

We define a relational contract as an equilibrium of this repeated game. An optimal relational contract is an equilibrium that maximizes the principal's payoff at the beginning of the game, i.e., π_1. As we are interested in the principal's optimal equilibrium, without loss of generality, we focus on the class of equilibria in which any observable deviation triggers the players to terminate the relationship, resulting in each player receiving their respective outside option payoffs.

4.2 Preliminary observations

As mentioned at the beginning of this section, we focus our analysis on the scenario where, in the baseline model, the agent has insufficient incentives to work because the manager is excessively aligned with the principal and does not recommend the project often enough (i.e., the region B in Figure 1). With a slight abuse of notation, we also denote this set of parameters as $B := \{(\alpha, \sigma) \mid \sigma \geq \sigma_0,$ and $(IC_{\mathcal{A}})$ is violated$\}$.

Consider a class of contracts where, in each period, the agent exerts effort ($e_t = 1$), and the project is implemented with probability η if the state and signal realization $(\omega, s) = (C, 0)$, but otherwise, it is implemented with certainty. Any such contract, if feasible, must satisfy the following incentive constraint of the agent: $\Pr(\iota_t = 1 \mid e_t = 1)b \geq c$, i.e.,

$$1 - \alpha\left(1 - \frac{1}{2}\sigma\right)(1 - \eta) \geq \frac{c}{b}. \qquad (IC^*_{\mathcal{A}})$$

Let η^* be the value of η for which $(IC^*_{\mathcal{A}})$ binds. (As the left-hand side of $(IC^*_{\mathcal{A}})$ increases in η, $(IC^*_{\mathcal{A}})$ holds for all $\eta \geq \eta^*$.) Fix $\eta = \eta^*$, and denote the agent's, the manager's, and the principal's payoff associated with such a contract as u^*, v^*, and π^*, respectively. As the agent is indifferent between exerting effort and shirking, $u^* = 0$.

Lemma 1. *Suppose $(\alpha, \sigma) \in B$. Then, the following holds:*

(i) $\pi^* \geq \pi_0$.

(ii) *When $v^* < v_0$, the only equilibrium is one in which all players take their outside options in all periods.*

(iii) *When $v^* \geq v_0$, in any equilibrium, the continuation payoffs of the manager and the principal at the beginning of each period are bounded above by v^* and π^*, respectively.*

Lemma 1 shows that for any equilibrium where the agent exerts effort, v^* and π^* are the upper bounds on the manager's and the principal's continuation payoff at each period. In particular, π^* is the maximal equilibrium payoff for the principal. Thus, any contract in the aforementioned class is indeed optimal if it is self-enforcing.

Note that any contract in this class requires the implementation of the project when $(\omega, s) = (C, 0)$ with some probability. However, when $(\omega, s) = (C, 0)$, the project is deemed "bad" by both the manager and the principal as their expected values are less than their outside options, and both would prefer to cancel it. Thus, a contract where such projects are implemented requires that either the manager or the principal sometimes take an action against their short-term interests so as to sharpen the agent's incentives to generate new projects.

In light of the above observation, one may consider two natural forms of relational contracts. First, it is the manager who takes an action against his short-term interests: in any given period, he recommends the implementation of a "bad" project with probability η^*, and the principal always adheres to the manager's recommendation. Second, it is the principal who takes such an action: in any given period, with probability η^*, she completely disregards the manager's report and implements any project that the agent may generate, but otherwise (with probability $1 - \eta^*$) she adheres to the manager's recommendation (which is identical to that in the static game).

We denote the former type of contracts as *mediation* contracts—the manager mediates between the principal and the agent by sometimes pooling the "bad" project with the rest. In contrast, we denote the latter type of contract as *intervention* contracts—the principal, with some probability, may intervene and implement the project by eschewing the manager's evaluation.

From Lemma 1 we know that whenever a mediation or an intervention contract is sustained in equilibrium, it is optimal. The next lemma shows that if neither of them can be sustained in equilibrium, then there is no other equilibrium with production. (In the Appendix, the proof of this lemma is presented after the proofs of Proposition 2 and Proposition 3 as it uses some conditions that are discussed later in the text).

Lemma 2. *Suppose* $(\alpha, \sigma) \in B$. *If neither the mediation contract nor the intervention contract can be sustained as an equilibrium, then the only equilibrium is one in which all players take the outside option in all periods.*

In what follows, we limit attention to these two types of contracts and analyze the conditions under which they are sustained in equilibrium.

5 On mediation and intervention contracts

5.1 Mediation contract

Under a *mediation* contract, the manager always reports $r = 1$ except when he sees a "bad" project, i.e., when $(\omega, s) = (C, 0)$. And if $(\omega, s) = (C, 0)$, he reports $r = 1$ if and only if the outcome of the randomizing device $z < \eta^*$, i.e., he reports $r = 1$ with probability η^*. Furthermore, the principal always follows the recommendation of the manager.

Recall that the manager would prefer to have a project canceled if he sees $(\omega, s) = (C, 0)$, and can indeed ensure cancellation of a "bad" project by reporting $r = 0$. But if the manager deviates and reports $r = 0$ when $z < \eta^*$, the agent triggers punishment by terminating the relationship. Thus, for such a contract to be feasible, the manager's on-path payoff in the game must be at least as large as his deviation payoff, which is his outside option. The following incentive constraint for the manager stipulates this condition:

$$(1-\delta)\Pr(V_{\mathcal{M}} = y \mid \omega = C, s = 0)y + \delta v^* \geq v_0,$$

i.e.,

$$(1-\delta)\left(\frac{1-\sigma}{2-\sigma}\right)y + \delta v^* \geq v_0. \tag{$IC_{\mathcal{M}}^{M}$}$$

On the other hand, the principal must have an incentive to implement the project when $r = 1$ even though such a message pools some unprofitable projects for her (when $s = 0$) with the profitable ones (when $s = 1$). Should she deviate, the manager triggers punishment by terminating the relationship, and the principal obtains her outside option. Therefore, the following incentive constraint must hold:

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid r = 1)Y + \delta\pi^* \geq \pi_0,$$

i.e.,

$$(1-\delta)\frac{\frac{1}{2}\sigma + \frac{1}{2}(1-\sigma)(1-\alpha(1-\eta^*))}{1-\alpha(1-\eta^*)\left(1-\frac{1}{2}\sigma\right)}Y + \delta\pi^* \geq \pi_0. \qquad (IC^M_{\mathcal{P}})$$

We can now characterize the conditions under which the mediation contract may be sustained as a PBE of the game. Note that from the definition of η^*, it follows that

$$\alpha_1(\sigma) = \alpha(1-\eta^*) = \frac{1-c/b}{1-\sigma/2}, \qquad (1)$$

where $\alpha_1(\sigma)$ is as defined in Proposition 1: for any $\sigma \in [\sigma_0, \sigma_1]$, $\alpha_1(\sigma)$ is the maximal loyalty level of the manager for which the agent still can be incentivized to exert effort in the static game (i.e., for a given σ, $(IC_{\mathcal{A}})$ binds at $\alpha_1(\sigma)$).

Proposition 2. **(*Mediation contract*)** *There exist $\delta_M \in (0,1)$ and $\alpha_M(\sigma,\delta) \geq \alpha_1(\sigma)$ such that for any $\delta \geq \delta_M$, the payoffs (v^*, π^*) can be sustained by the mediation contract if $(\alpha,\sigma) \in B_M := \{(\alpha,\sigma) \mid \alpha_1(\sigma) \leq \alpha \leq \alpha_M(\sigma,\delta), \sigma \geq \sigma_0\}$. In particular, the following holds for the mediation contract:*

- (i) *When $(\omega, s) = (C, 0)$, the manager reports $r = 0$ with probability $1-\eta^*$ and reports $r = 1$ with probability η^*; otherwise, the manager reports $r = 1$. The principal implements the project if and only if $r = 1$.*
- (ii) *The threshold $\alpha_M(\sigma,\delta)$ increases in δ but may be increasing or decreasing in σ, depending on the value of δ:*
  - (a) *If $\delta_M < \hat{\delta} := b/(2b-c)$, then $\alpha_M(\sigma,\delta)$ decreases in σ if $\delta_M \leq \delta \leq \hat{\delta}$ and increases in σ if $\delta \geq \hat{\delta}$.*
  - (b) *If $\delta_M > \hat{\delta}$, $\alpha_M(\sigma,\delta)$ always increases in σ for all $\delta \geq \delta_M$.*

Proposition 2 shows that when players are sufficiently patient, the mediation contract may indeed be the optimal relational contract provided that the manager is not too loyal to the principal. The threshold on the loyalty, α_M, is increasing in the discount rate δ. The impact of the manager's skill level (σ) on the loyalty threshold is more nuanced: the threshold decreases in σ if δ is low but increases in σ otherwise.

To see the intuition for this result, recall that we are focusing on a scenario where $(\alpha,\sigma) \in B$; i.e, in the static game, the agent's incentive constraint $(IC_{\mathcal{A}})$ is violated and no project gets generated. The agent correctly anticipates that the manager, given his skill and loyalty, would be too discerning in his recommendation for project implementation, and the agent's expected benefit from the project would not cover his cost of effort. To encourage the agent to exert effort, a mediation contract props up the likelihood of project implementation as the manager recommends some projects that he would otherwise prefer to cancel. The manager is expected to behave as if his loyalty to the principal, α, were lowered to $\alpha_1(\sigma)$. In other words, the relational incentives drive a

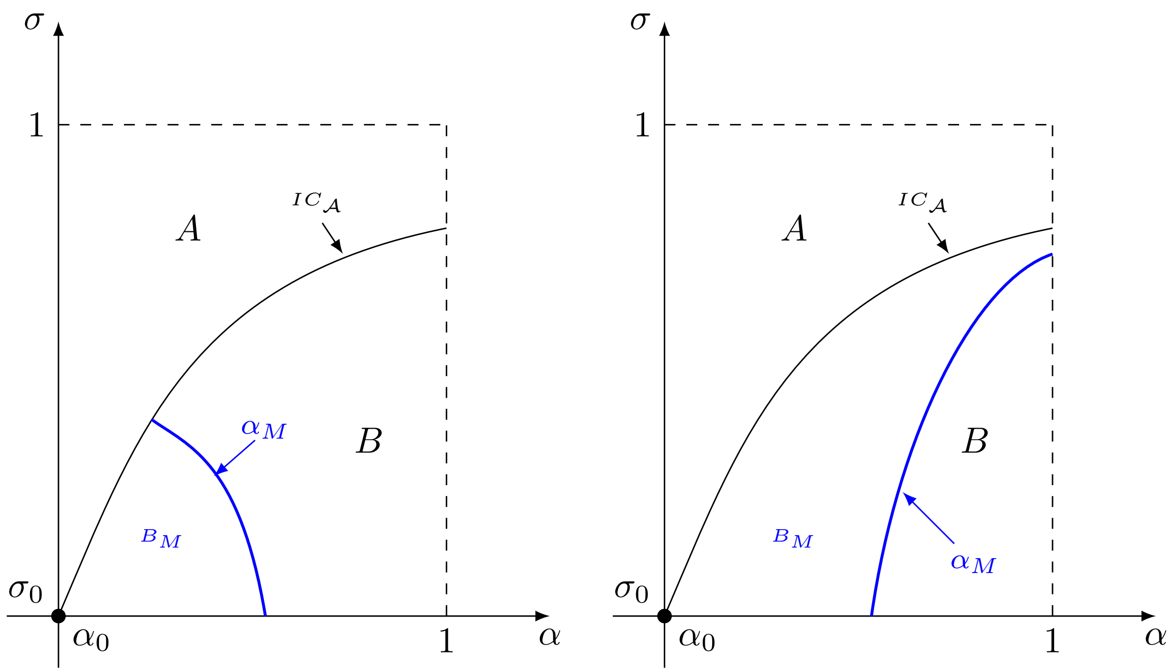

Figure 3: Mediation can be sustained as the optimal relation contract in region B_M.
Left panel: $\delta_M < \hat{\delta}$ and $\delta \in (\delta_M, \hat{\delta})$; right panel: $\delta > max\{\hat{\delta}, \delta_M\}$.

wedge between the manager's real loyalty (α) and his "revealed" loyalty $(\alpha_1(\sigma))$ to the principal when it comes to reporting on the project.

An immediate implication of this observation is that the principal's incentive constraint, $(IC^M_{\mathcal{P}})$, is always satisfied: under such a contract, the principal's payoff in each period is the same as her payoff in the static model if the manager's loyalty were $\alpha_1(\sigma)$, and, by $(IC_{\mathcal{P}})$, for any $\sigma > \sigma_0$ this payoff is at least π_0 (i.e., the point $(\alpha_1(\sigma), \sigma)$ lies on the boundary of region B where $(IC_{\mathcal{P}})$ is always satisfied). Thus, a *mediation* contract can be supported in a PBE as long as it satisfies the manager's incentive constraint $(IC^M_{\mathcal{M}})$.

Consider the manager's on-path payoff as given by the left-hand side of $(IC^M_{\mathcal{M}})$. One may readily note that for any $(\alpha, \sigma) \in B$ that is sufficiently close to $(IC_{\mathcal{A}})$, the associated value of η^* is also small. Hence, the manager's continuation payoff, v^*, is sufficiently close to his static game payoff, which is always larger than his outside option v_0. Thus, for any such $(\alpha, \sigma) \in B$, the manager's continuation payoff $v^* > v_0$, and therefore, $(IC^M_{\mathcal{M}})$ can be met if δ is sufficiently large. Conversely, for any δ large enough, there exists a set of (α, σ) that satisfies this constraint. To characterize this set, we need to explore how these parameters affect the manager's current and continuation payoffs.

The manager's current period payoff (when he recommends a bad project) decreases in σ but is independent of α. The signal $s = 0$ is a stronger indication of low value $(V_{\mathcal{M}} = 0)$ when the signal is more precise (i.e., when σ is high). But as the manager's current period payoff reflects his expected value of the project conditional on the state being C, it is not affected by the prior distribution (α) over the states.

The manager's continuation payoff, v^*, however, is affected by both σ and α: it is increasing in σ and decreasing in α. A higher precision of the signal means that ex-ante, the manager is more likely to get his preferred projects implemented and it is less likely that a project gets implemented when he prefers to have it canceled. That is, a higher σ moves probability from the state-signal pair $(\omega, s) = (C, 0)$ where the manager gets 0 from the project to the other three pairs where the manager gets a benefit of y. But a higher loyalty, α, moves probability from state D to state C. In particular, the ex-ante likelihood of obtaining the state-signal pair $(\omega, s) = (C, 0)$ increases and that of $(\omega, s) = (D, 0)$ decreases. That is, it becomes more likely that the manager would find himself in a scenario where he prefers to cancel the project but it still gets implemented, and it becomes less likely that a project that may be deemed valuable to the manager would be implemented.[6]

As α always reduces the manager's on-path payoff, we readily see that for any given σ and (large enough) δ, there is a cutoff value of α, α_M, for which ($IC^M_{\mathcal{M}}$) binds and is violated if and only if $\alpha > \alpha_M$. To see how α_M depends on σ, note that since σ has opposing effects on the manager's current and continuation payoffs, its net effect would depend on δ, the relative weight on the two payoffs. When δ is relatively small, σ's negative effect on the current payoff dominates its positive effect on the continuation payoff, and the manager's on-path payoff decreases in σ. Consequently, for ($IC^M_{\mathcal{M}}$) to bind, α_M must fall. But the opposite holds when δ is sufficiently large. As the manager's on-path payoff now increases in σ, for ($IC^M_{\mathcal{M}}$) to bind, α_M must increase as well.

5.2 Intervention contract

We now explore the feasibility of an *intervention* contract. Under the intervention contract, the principal may sometimes decide on implementation by completely ignoring the manager's report. In particular, if the outcome of the public randomizing device $x < \eta^*$, she always implements the project disregarding the manager's recommendation, and anticipating that, the manager sends a completely uninformative report. If instead $x \geq \eta^*$, the principal and the manager behave as in the static game: the manager reports $r = 0$ when $(\omega, s) = (C, 0)$ and reports $r = 1$ otherwise, and the principal implements the project if and only if $r = 1$.

Notice that in an intervention contract, the manager does not have any profitable deviation from his reporting strategy. In the case where the principal is expected to ignore the manager's report, he cannot do any better than babbling. And in the case where the principal adheres to the manager's report, the manager's equilibrium reporting strategy calls for the implementation of the project only when it is indeed profitable for the manager to do so. Thus, we only have to ensure that the manager has the incentive to participate in this contract, i.e.,

$$v^* \geq v_0. \qquad (IC^I_{\mathcal{M}})$$

[6] Note that when $s = 1$ the project is always valuable and is surely implemented, but the probability of obtaining this signal does not depend on α.

The principal, on the other hand, may be tempted to cancel a project when he is expected to proceed with it. There are two cases to consider. First, if $x < \eta^*$ the principal must implement any project that the agent generates even though the project's (prior) expected value is less than the principal's outside option. Thus, the principal's on-path payoff must be larger than his deviation payoff:

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y)Y + \delta\pi^* = (1-\delta)\frac{1}{2}Y + \delta\pi^* \geq \pi_0. \qquad (IC^I_{\mathcal{P}})$$

Second, when $x \geq \eta^*$, the principal must have incentives to adhere to the manager's recommendation who calls for implementation except when $(\omega, s) = (C, 0)$. Thus, we have the following constraint:

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid (\omega, s) \neq (C, 0))Y + \delta\pi^* \geq \pi_0. \qquad (IC^{I'}_{\mathcal{P}})$$

One may, however, readily note that ($IC^I_{\mathcal{P}}$) implies ($IC^{I'}_{\mathcal{P}}$), and hence, ($IC^{I'}_{\mathcal{P}}$) can be ignored in our subsequent analysis. The proposition below characterizes the conditions under which an intervention contract may be sustained as a PBE of the game.

Proposition 3. **(*Intervention contract*)** *There exist $\delta_I \in (0,1)$, and cutoffs $\alpha_I(\sigma) \in [\alpha_1(\sigma), 1]$ and $\sigma_I(\delta) \in [\sigma_0, \sigma_1]$ such that for any $\delta \geq \delta_I$, the payoff (v^*, π^*) can be sustained by the intervention contract if $(\alpha, \sigma) \in B_I := \{(\alpha, \sigma) \mid \alpha_1(\sigma) \leq \alpha \leq \alpha_I(\sigma), \sigma_I(\delta) \leq \sigma \leq \sigma_1\}$. In particular, the following holds for the intervention contract:*

(*i*) *If $x \leq \eta^*$, the principal implements any project that the agent generates and the manager sends a report that is completely uninformative. Otherwise, the manager reports $r = 0$ when $(\omega, s) = (C, 0)$ and reports $r = 1$ otherwise. The principal implements the project if and only if $r = 1$.*

(*ii*) *The cutoff $\alpha_I(\sigma)$ increases in σ and the cutoff $\sigma_I(\delta)$ decreases in δ; furthermore, $\sigma_I(\delta_I) = \sigma_1$ and $\sigma_I(\delta) = \sigma_0$ as $\delta \to 1$.*

Proposition 3 closely parallels its counterpart on mediation (Proposition 2). When the players are sufficiently patient, intervention is an optimal relational contract provided that the manager is not too loyal to the principal but is sufficiently skilled. The skill threshold decreases in δ, whereas the loyalty threshold α_I is independent of δ but increases in σ.

The intuition for this result also bears close resemblance to that of Proposition 2, though there are a few subtle differences. First, the principal's incentive constraint ($IC^I_{\mathcal{P}}$) may bind as the principal would prefer to reject any project that the agent may generate when she lacks further information about its quality. As she implements the project ignoring the manager's report, the precision of the manager's signal, σ, does not affect her current period payoff whereas her continuation payoff is increasing in σ. Thus, the left-hand side of ($IC^I_{\mathcal{P}}$) increases with σ. Furthermore, the principal's continuation payoff π^* is independent of α as the state affects the principal's payoff only through the ex-ante probability of project implementation, which remains constant when ($IC^*_{\mathcal{A}}$) binds (i.e., η^*, the probability of implementation through principal's intervention, adjusts with α

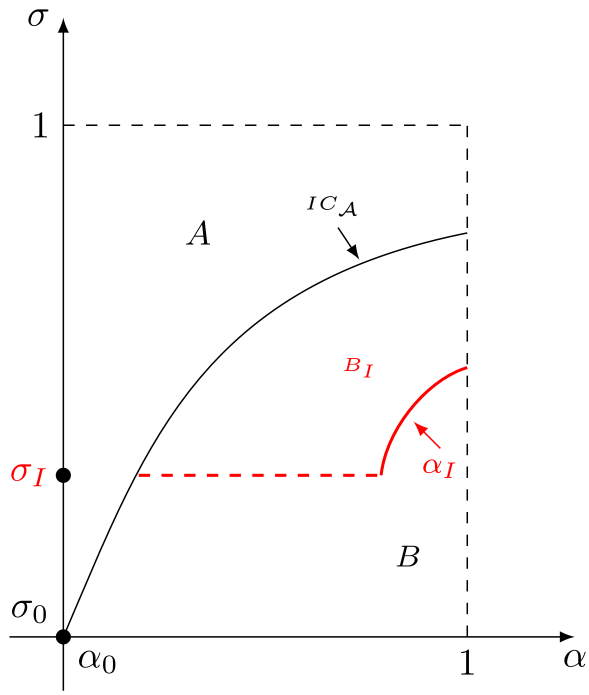

Figure 4: Intervention can be sustained as the optimal relation contract in region B_I.

to ensure $\alpha(1-\eta^*)$ stays fixed at $\alpha_1(\sigma)$). Thus, the principal's on-path payoff, the left-hand side of ($IC^I_{\mathcal{P}}$) is increasing in σ but unaffected by α.

This observation, combined with the fact that the on-path payoff is also increasing in δ, implies that for any δ not too low, ($IC^I_{\mathcal{P}}$) would be satisfied if σ is above a threshold $\sigma_I(\delta)$, and this threshold decreases with δ. Thus, the intervention contract remains feasible as long as it ensures the manager's participation, i.e., it satisfies ($IC^I_{\mathcal{M}}$). And as the manager's continuation value v^* is decreasing in α, there is a threshold α_I (given σ) such that ($IC^I_{\mathcal{M}}$) is satisfied if and only if $\alpha \leq \alpha_I(\sigma)$.

That $\alpha_I(\sigma)$ increases in σ can be gleaned from the fact that $\alpha_I(\sigma)$ is the value of α (given σ) for which ($IC^I_{\mathcal{M}}$) binds (see Lemma 1), and v^* is increasing in σ and decreasing in α. As v^* is independent of δ, so is α_I.

6 Optimal relational contract

In light of the mediation and intervention contracts discussed above, we can now explore how the manager's characteristics affect the optimal relational contract.

Theorem 1. (***Optimal relational contract***) *Fix Y/π_0, and let $\delta \geq \delta_I$ so that $B_I \neq \emptyset$. Then, there exist two cutoffs τ_1 and τ_2 ($> \tau_1$) such that:*

(i) If $y/v_0 < \tau_1$, then $B_M = \emptyset$, i.e., the mediation contract cannot be sustained.

(*ii*) *If* $\tau_1 \leq y/v_0 < \tau_2$*, then* $B_M \setminus B_I \neq \emptyset$ *and* $B_I \setminus B_M \neq \emptyset$*, i.e., there exist parameter regions where the optimal contract is sustained only by the mediation contract and parameter regions where the optimal contract is sustained only by the intervention contract.*

(*iii*) *If* $y/v_0 \geq \tau_2$*, then* $B_I \subseteq B_M$*, i.e., if the optimal relational contract can be sustained by the intervention contract, it can also be sustained by the mediation contract.*

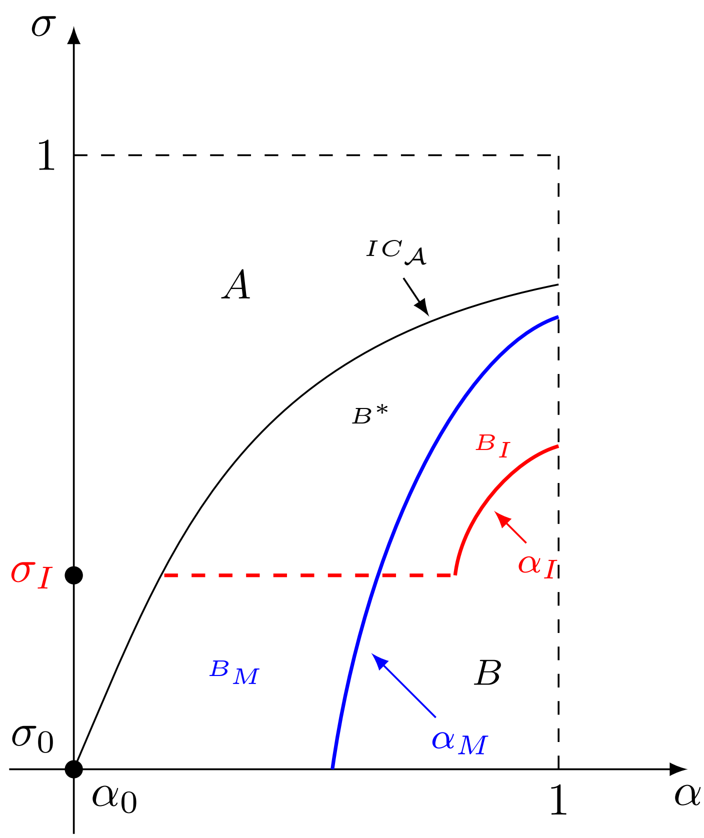

Figure 5: Mode of communication for high δ (i.e., $\delta > max\{\delta_M, \delta_I\}$): In region B_M only mediation is feasible; in B_I only intervention is feasible; in region B^* both modes are feasible.

The above theorem indicates that the optimality of the mediation contract critically hinges on the ratio of the manager's project value to his outside option (y/v_0). Only the intervention contract may be sustained when this ratio is sufficiently small (part (i)). In contrast, when this ratio is sufficiently large, mediation dominates intervention—if intervention is optimal then so is mediation, but mediation could be sustained as a PBE even when the intervention contract remains infeasible (part (iii)). However, the most interesting case arises when this ratio is at an intermediate range. Here, depending on the parameter values, either the intervention or the mediation contract may emerge as the *unique* optimal relational contract (part (ii); Figure 5 illustrates this case).

To see the intuition for this result, consider part (ii) first. Recall that the manager's continuation value of the relationship (v^*) is increasing in his skill but decreasing in loyalty, whereas the principal's continuation value (π^*) increases in the manager's skill (σ) but does not depend on his loyalty (α). When the manager is not very skilled, the principal's continuation value of the relationship is low, but the manager's continuation value may still be large provided he is not quite loyal to the principal. Thus, the threat of termination by the agent may deter the manager from deviating but would not deter the principal. Consequently, only the mediation contract may be sustained. But for a highly skilled manager with high loyalty to the principal, the opposite holds. His continuation value may be too low to deter deviation whereas the principal may have a high continuation value and would not find it profitable to deviate.[7] Thus, when the manager's skill (σ) and loyalty (α) are not too high, mediation may be the unique optimum, and when the manager is highly skilled and relatively loyal, intervention may be the unique optimum.

This result shows why and how the mode of communication and decision-making in an organization may be driven by the characteristics of its middle managers. If the manager's skill and loyalty are both relatively low, mediation is optimal. In equilibrium, while recommending implementation, the manager sometimes pools a "bad" project (one that is likely to yield low value to both the manager and the principal) with "good" ones (one that is sure to yield high value to the manager), and the principal always accepts the manager's recommendation. In contrast, if the manager's skill and loyalty are relatively high, an intervention contract becomes optimal. In such a contract, some of the time, the principal intervenes and directly implements the project, ignoring the manager's report. And on other times, the manager recommends only those projects that are profitable to him, and the principal abides by his recommendation.

Following the seminal work by Bertrand and Schoar (2003), a large literature has documented a strong influence of manager fixed effects in corporate practices or "management styles"—i.e., the pattern of decision-making —observed across organizations. The variations in styles are typically attributed to the variations in the top management's preferences, skills, or biases in their opinions (Rotemberg and Saloner, 1993; Bertrand and Schoar, 2003; Dessein and Santos, 2021). An important implication of this finding is that the management style that the headquarters adopts could stem from the personal characteristics of the middle managers rather than that of the CEO herself.

To see the intuition for parts (*i*) and (*iii*), it is useful to make the following observation in relation to the feasibility of the mediation contract (as given in Proposition 2).

Lemma 3. *δ_M is decreasing in y/v_0 and $\alpha_M(\sigma, \delta)$ is increasing in y/v_0.*

The larger the project value to outside option ratio for the manager, the more he is likely to gain from continuing the relationship, and the smaller is his temptation to renege on the mediation contract. Thus, δ_M, the minimum value of δ that sustains the mediation contract, is decreasing in y/v_0. This observation readily implies the result in part (*i*): given δ, if y/v_0 is sufficiently small, the mediation contract is not sustainable. Finally, recall that the mediation contract is sustainable if and only if $\alpha \leq \alpha_M(\sigma, \delta)$. As noted above, when the middle manager gains more from continuing

[7] It is also worthwhile to note that $B_M = B_I = \{(\alpha, \sigma) \mid \alpha_1(\sigma) \leq \alpha \leq \alpha_I(\sigma), \sigma_0 \leq \sigma \leq \sigma_1\}$ as $\delta \to 1$. As the players put little weight on their deviation gains in the current period, both mediation and intervention contracts can be sustained as long as the players' continuation payoffs exceed their respective outside options, i.e., $(\pi^*, v^*) \geq (\pi_0, v_0)$.

the relationship, he is more willing to follow the mediation contract. Hence, as stated in part (*iii*), an increase in y/v_0 increases the threshold $\alpha_M(\sigma, \delta)$, expanding the range where the mediation contract is optimal.

The above result highlights the role of the manager's loyalty *to the organization* as captured by his project value to outside option ratio in the mediation contract. In particular, the manager can successfully mediate between the bottom and the top rungs of the organization only if he is sufficiently loyal to the organization. Furthermore, in the optimal mediation contract, his loyalty to the organization complements his loyalty to the top: a manager who is keenly loyal to the top can still succeed in his dual role (of incentivizing the workers at the bottom while maintaining credibility at the top) if he is also sufficiently loyal to the organization.

We conclude this section with the following two remarks. First, in our model of hierarchical organization, a relational contract consists of three bilateral but interconnected relations among the agent, manager, and principal. When the mediation contract is optimal, it is the relational contract between the manager and the agent that ensures the provision of effort incentives, whereas, under the intervention contract, the relational contract between the principal and the agent comes into play. Thus, our finding highlights that depending on the characteristics of the manager, the provision of incentives may rely on different bilateral relations within the organization.

Second, recall that for the mediation contract to be feasible, the principal may not observe the realization of the randomizing device z, as otherwise, it need not be sequentially rational for the principal to always follow the manager's recommendation. Thus, our result indicates that it may be beneficial for the principal to keep the relationship between the agent and the manager at arm's length. The principal may undermine incentives in the organization by micromanaging the relational contract between the agent and the manager where the principal demands to see the realization of z. In particular, we have the following corollary.

Corollary 1. *Suppose that the randomizing device z is public. Then, for any $(\alpha, \sigma) \in B \backslash B_I$, the only equilibrium is one in which all players take the outside option in all periods.*

That is, when z is public, if the optimal relational contract can be sustained by the mediation contract, it can also be sustained by the intervention contract.

7 Discussion and conclusion

7.1 On discouraging implementation

Our analysis of the baseline model shows that the middle manager may also lose his effectiveness when he is too close to the agent as he recommends implementation too frequently. This type of distortion arises in regions A and D in Figure 1. To ameliorate this problem in a long-term relationship, the parties must discipline the manager to recommend projects in a way that is more aligned with the principal's interests.

The key, then, is to incentivize the manager to forgo some of the projects that bring him private benefits but are unprofitable for the principal (i.e., projects with $\omega = D$ and $s = 0$).[8] Since the principal's realized value from the project (when implemented), $V_{\mathcal{P}}$, is publicly observed, the principal can discipline the manager by threatening to terminate the relationship if the manager's recommended project results in a poor outcome. However, the principal may need to exercise some leniency to prevent the manager from becoming too cautious in making recommendations, which could blunt the agent's incentives to exert effort. To strike this balance, the punishment for a poor outcome may be conditioned on the realization of the public randomization device, x.

In what follows, we call an action profile of the stage game "*β-lenient*" if the agent exerts effort, the principal follows the manager's recommendation, and the manager reports $r = 1$ if $s = 1$, $r = 0$ if $(\omega, s) = (C, 0)$, and when $(\omega, s) = (D, 0)$ he reports $r = 1$ if $x \leq \beta$ and $r = 0$ if $x > \beta$. The principal terminates the relationship only if $V_{\mathcal{P}} = 0$ and $x > \beta$. That is, with probability β, the manager may recommend a "bad" project (i.e., $s = 0$) as he is assured of leniency should the project yield a poor outcome for the principal. Note that, when $\beta = 1$, the pattern of project implementation coincides with that in the baseline model (Proposition 1); and when $\beta = 0$, the action profile indicates that only the "good" project (i.e., $s = 1$) can get implemented.

Proposition 4. *Suppose* $(\alpha, \sigma) \in \{(\alpha, \sigma) \mid \sigma \geq \sigma_0$ *and* $\alpha < \alpha_1(\sigma)\}$.

1. *There exist* $\bar{\delta}$ *(which depends on* α *and* σ*) such that if* $\delta \in [\bar{\delta}, 1)$*, the stationary contract, where the parties' action profile is* β*-lenient in every period (for some* $\beta \in [0, 1]$*), is optimal.*

2. *If* $\sigma > \sigma_0$*, then there exists* $\underline{\delta} \in (0, \bar{\delta})$ *(which depends on* α *and* σ*) such that if* $\delta \in [\underline{\delta}, \bar{\delta})$*, the optimal relational contract is non-stationary and can be the following:*

   (*a*) *In the first period, the parties start the relationship by playing the* β*-lenient action.*

   (*b*) *In the subsequent periods, their action randomizes between* β*-leniency and* $\beta'-$*leniency for some* $\beta' \in (\beta, 1]$.

   (*c*) *Once the* $\beta'-$*lenient action is realized, the future actions are always* β'*-lenient.*

(The proof of the above proposition relies on techniques that closely resemble the methods developed for the characterization of the PPE payoff as in Abreu et al. (1990), and is given in the Online Appendix.) Proposition 4 shows that the optimal relational contract allows for leniency while using termination for poor project outcome, and the contract dynamics depend on the players' discount factor. Recall that in Regions A and D, any improvement over the static equilibrium requires the manager to sometimes ignore his short-term private benefit and recommend implementation in line with the principal's interest. In the dynamic relationship, the principal fulfills the requirement by promising the manager a high enough continuation payoff if he obeys and by

[8] In region D, the manager can mediate between the principal and the agent by using the randomizing device z_t, as is the case in the mediation contract discussed in Section 5.1. Since the manager can pool the "bad" projects (those with $(\omega, s) = (D, 0)$) with the "good" projects (those with $s = 1$) without the principal being aware, the principal's incentive constraint will be trivially satisfied. Hence, we can focus on the manager's incentives.

threatening him with termination if he deviates. Since the principal's optimal action (among the individually rational actions) can provide the manager with a payoff strictly higher than v_0, such an action can be sustained as an equilibrium as long as δ is sufficiently close to one.

However, when δ is not sufficiently high, sustaining the principal's optimal action on the equilibrium path counts on back-loading the incentives for the manager. Specifically, the principal must promise the manager higher future leniency to enforce more stringent actions in the earlier periods, while maintaining the agent's incentive to exert effort. This can be fulfilled as long as the manager's skill is above σ_0. In the long run, the relationship falls into the action with relatively higher leniency. Such back-loaded incentives and long-run dynamics are reminiscent of the optimal contracts explored in the recent literature on relational incentives with private information (see, e.g., Ray, 2002, Chassang, 2010, and Li et al., 2017).

7.2 An alternative information structure for the manager's signal

In our model, the middle manager always identifies a bad project for the principal with certainty but detects a good project only some of the time. Thus, the "skill" of the manager refers to his ability to identify a good project as such. Such a "good news is definitive" feature of the signal process helps in improving the analytical tractability of the model, but it is not essential for delivering its key insights on the value of "moderate" loyalty and how the range of moderation expands with the manager's skill (Proposition 1).

To see this, consider the following alteration to our baseline model: as before, assume that once a project is generated the manager observes its type (ω) and the signal s on its value to the principal, but the informativeness of the manager's signal is given now as:

$$\Pr(s = 1 \mid V_{\mathcal{P}} = Y) = \Pr(s = 0 \mid V_{\mathcal{P}} = 0) = \sigma > \frac{1}{2}.$$

That is, neither of the two signals is definitive but $s = 1$ is better news about the project than $s = 0$ with σ being the precision of the signal. We continue to focus on a parameter region where informative communication can be supported in equilibrium, and, analogous to Assumption 1, we impose the following parametric restrictions:

Assumption 1′. (i) $(1 - \sigma)y < v_0 < \sigma y$ and (ii) $(1 - \sigma)Y < \pi_0 < \sigma Y$.

As in our baseline model, in any equilibrium, the manager will always recommend the project when $\omega = D$ or $(\omega, s) = (C, 1)$, as he obtains an expected project value of y or σy, respectively, both exceeding his outside option v_0 (by Assumption 1′). Thus, in any equilibrium where the manager's report is informative, it must be that $r = 0$ if and only if $(\omega, s) = (C, 0)$. And for such a reporting strategy to be sequentially rational for the manager, it must be that

$$v_0 > \mathbb{E}[V_{\mathcal{M}} \mid (\omega, s) = (C, 0)] = (1 - \sigma)y.$$

which also holds under Assumption 1′.

Next, consider the incentive constraints of the principal that ensure that she follows the manager's recommendation: $\Pr\left(V_{\mathcal{P}}=Y \mid r=0\right) Y$
$<\pi_0 \leq \Pr\left(V_{\mathcal{P}}=Y \mid r=0\right) Y$, i.e.,

$$(1-\sigma)Y < \pi_0 \leq \frac{1-\alpha(1-\sigma)}{2-\alpha}Y, \qquad (IC_{\mathcal{P}}^*)$$

and the constraint on the agent that ensures he exerts effort to generate a project, i.e., $\Pr\left(r=1 \mid e=1\right) b \geq c$, i.e.,

$$\left(1-\frac{1}{2}\alpha\right) b \geq c. \qquad (IC_{\mathcal{A}}^*)$$

Thus, a PBE with informative communication exists if and only if ($IC_{\mathcal{P}}^*$) and ($IC_{\mathcal{A}}^*$) hold, and these two constraints stipulate a lower and upper bound on α (observe that under Assumption 1', only the right-hand side inequality in ($IC_{\mathcal{P}}^*$) may be binding). Hence, we again obtain the result that for a middle manager to be effective, his loyalty to the top must lie in a moderate range. Moreover, as noted in our baseline model, this range is increasing in the manager's skill as ($IC_{\mathcal{P}}^*$) is relaxed when σ increases.

7.3 Conclusion

Middle managers undertake a dual role that is particularly critical in innovative organizations: as a conduit of information flow between the headquarters and the workers "on the ground," they must credibly convey decision-relevant information to the top but also keep the junior workers motivated by promoting their innovative ideas and securing corporate support for their work. Thus, a middle manager needs to be skilled in screening good ideas from the bad ones and must navigate the delicate balance of dual loyalty, maintaining credibility with both the top and bottom rungs of the organization.

Our analysis illustrates how these personal characteristics of the middle managers—his loyalty to the top (relative to his subordinates) and his skill—drive his effectiveness on the job. Furthermore, in long-term employment relationships, loyalty may be managed through strategic communication and decision-making protocols, improving the efficacy of a middle manager. In turn, our findings highlight how the locus of the middle manager's loyalty within the organization interacts with his loyalty towards the organization, shapes the nature of incentive provision within the organization, and influence the management style (i.e., the decision-making pattern) of its leadership.

Appendix

This Appendix contains the proofs omitted in the text.

Proof of Proposition 1. **Step 1.** Consider any equilibrium in which the principal's project implementation policy depends on the manager's report r, i.e., an equilibrium in which $\Pr(\iota=1 \mid r=$

1) $\neq \Pr(\iota = 1 \mid r = 0)$. Suppose, without loss of generality, that $\Pr(\iota = 1 \mid r = 1) > \Pr(\iota = 1 \mid r = 0)$. Since

$$\mathbb{E}[V_{\mathcal{M}} \mid (\omega, s)] = y > v_0 \text{ for all } (\omega, s) \neq (C, 0),$$

in such equilibrium the manager necessarily reports $r = 1$ whenever $(\omega, s) \neq (C, 0)$. Similarly, since

$$\mathbb{E}[V_{\mathcal{M}} \mid (C, 0)] = ((1-\sigma)/(2-\sigma))y < v_0,$$

the manager necessarily reports $r = 0$ when $(\omega, s) = (C, 0)$.

Step 2. Given the above reporting strategy, it is optimal for the principal to follow the manager's recommendation—i.e., the project implementation policy $\Pr(\iota = 1 \mid r = 1) = \Pr(\iota = 0 \mid r = 0) = 1$ is optimal—whenever ($IC_{\mathcal{P}}$) and ($IC'_{\mathcal{P}}$) are satisfied. And, given the above reporting and project implementation strategies, it is optimal for the agent to choose $e = 1$ when ($IC_{\mathcal{A}}$) is satisfied. Thus, when ($IC_{\mathcal{P}}$), ($IC'_{\mathcal{P}}$) and ($IC_{\mathcal{A}}$) are satisfied, there exists an equilibrium in which $e = 1$, the manager recommends the project's implementation if $(\omega, s) \neq (C, 0)$ and cancellation if $(\omega, s) = (C, 0)$, and the principal follows the manager's recommendation, provided the principal's ex-ante payoff from this equilibrium is at least as large as his outside option.

Step 3. As noted in the text, ($IC'_{\mathcal{P}}$) is always satisfied under Assumption 1(i). Hence, such an equilibrium exists if and only if ($IC_{\mathcal{P}}$) and ($IC_{\mathcal{A}}$) are satisfied and the principal's payoff exceeds π_0. Observe that ($IC_{\mathcal{P}}$) is equivalent to

$$\alpha \geq \frac{2\pi_0 - Y}{\sigma\,(Y - \pi_0) + (2\pi_0 - Y)} := \alpha_0,$$

and ($IC_{\mathcal{A}}$) is equivalent to

$$\alpha \leq \frac{1 - \frac{c}{b}}{1 - \frac{1}{2}\sigma}.$$

Let $\alpha_1 := \min\{(1 - \frac{c}{b})/\left(1 - \frac{1}{2}\sigma\right), 1\}$. It is routine to check that $\alpha_1 \geq \alpha_0$ if and only if

$$\sigma \geq \sigma_0 := \frac{2c(2\pi_0 - Y)}{2c(\pi_0 - Y) + Yb},$$

and that $\alpha_1 = \alpha_0$ if $\sigma = \sigma_0$.

Step 4. (*Proof of Part* (*i*)) When $\sigma < \sigma_0$, either ($IC_{\mathcal{P}}$) or ($IC_{\mathcal{A}}$) is violated. If ($IC_{\mathcal{P}}$) is violated, there is no equilibrium where the principal's implementation policy depends on the manager's report. In any equilibrium in which the project implementation is independent of the manager's report, the project is always rejected. So, $e = 0$ and all parties obtain their outside options. Similarly, when ($IC_{\mathcal{A}}$) is not satisfied, in all equilibria, $e = 0$ and all parties obtain their outside options. In both cases, the principal's payoff is π_0.

Step 5. (*Proof of Part* (*ii*)) For any $\sigma \geq \sigma_0$ and $\alpha \in [\alpha_0, \alpha_1]$ the principal's ex-ante payoff under the aforementioned strategy profile (i.e., $e = 1$, the manager reports cancellation if and only if $(\omega, s) = (C, 0)$, and the principal always follows the manager's recommendation) is

$$\pi = \Pr(r = 1)\mathbb{E}[V_{\mathcal{P}} \mid r = 1] + \Pr(r = 0)\pi_0 \geq \pi_0$$

(the inequality follows from the fact that the $(IC_{\mathcal{P}})$ implies that $\mathbb{E}[V_{\mathcal{P}} \mid r = 1] \geq \pi_0$). Thus, the above strategy profile constitutes an equilibrium of the game.

If $\sigma \geq \sigma_0$ but $\alpha \notin [\alpha_0, \alpha_1]$ either $(IC_{\mathcal{P}})$ or $(IC_{\mathcal{A}})$ is violated, and as argued in Step 4 above, an equilibrium where $e = 1$ there does not exist, and the parties take their outside option.

Finally, it is routine to check from the formula for α_0 and α_1 given in Step 3 that α_0 decreases in σ whereas α_1 increases in σ with $\alpha_1 = 1$ if $\sigma \geq \sigma_1 := 2c/b$.

□

Proof of Lemma 1. **Step 1**. Proof of Part (i): We begin by showing that $\pi^* \geq \pi_0$. Decompose π^* as

$$\begin{aligned}\pi^* &= \pi_0 + \Pr(r = 1)(\mathbb{E}(V_{\mathcal{P}} \mid r = 1) - \pi_0) \\ &= \pi_0 + \Pr(r = 1)(\Pr(V_{\mathcal{P}} = Y \mid r = 1)Y - \pi_0) \\ &= \pi_0 + \Pr(s = 1)\Pr(r = 1 \mid s = 1)(\Pr(V_{\mathcal{P}} = Y \mid s = 1)Y - \pi_0) + \\ &\qquad \Pr(s = 0)\Pr(r = 1 \mid s = 0)(\Pr(V_{\mathcal{P}} = Y \mid s = 0)Y - \pi_0) \\ &= \pi_0 + \frac{1}{2}\sigma(Y - \pi_0) + \left(1 - \frac{1}{2}\sigma\right)(1 - \alpha(1 - \eta^*))\left(\frac{1 - \sigma}{2 - \sigma}Y - \pi_0\right).\end{aligned} \tag{2}$$

Now, from (1) we have $\alpha(1 - \eta^*) = \alpha_1$, and recall that $\alpha_1 \geq \alpha_0$ where α_1 and α_0 are as defined in Proposition 1. From these two observations and (2), it follows that

$$\begin{aligned}\pi^* &\geq \pi_0 + \frac{1}{2}\sigma(Y - \pi_0) + \left(1 - \frac{1}{2}\sigma\right)(1 - \alpha_0)\left(\frac{1 - \sigma}{2 - \sigma}Y - \pi_0\right) \\ &= \pi_0,\end{aligned}$$

where the equality is obtained by replacing α_0 with its expression as given in the proof of Proposition 1 (recall that by definition, at $\alpha = \alpha_0$, $(IC_{\mathcal{P}})$ binds, i.e., $\Pr(V_{\mathcal{P}} = Y \mid r = 1)Y - \pi_0 = 0$).

Step 2. *Proof of Part (ii) and the first result in Part (iii)*:

Step 2a. We first claim that for any equilibrium, it is without loss to compute the manager's continuation payoff at any period t conditional on the agent's history $(h^{\mathcal{A},t})$ rather than the manager's $(h^{\mathcal{M},t})$. Formally, consider an arbitrary equilibrium, and fix an arbitrary period t. Recall that the private history of the agent up to date t is contained in that of the manager's. In what follows, with a slight abuse of notation, we denote this fact as $h^{\mathcal{A},t} \subset h^{\mathcal{M},t}$. Our claim can be represented as $v_t(h^{\mathcal{M},t}) = v_t(h^{\mathcal{A},t})$.

The claim follows from the following two observations. First, because the projects are i.i.d across periods, the manager's histories may affect his payoff only through their effect on his strategy. Second, note that the agent's strategy only depends on his private history $h^{\mathcal{A},t}$, the principal's strategy only depends on the public history $h^{\mathcal{P},t}$, and both $h^{\mathcal{A},t}$ and $h^{\mathcal{P},t}$ are part of the manager's history $h^{\mathcal{M},t}$ (as for any given period τ, $h^{\mathcal{P},\tau} \subset h^{\mathcal{A},\tau} \subset h^{\mathcal{M},\tau}$). Hence, given $h^{\mathcal{M},t}$, the manager's best response depends on $h^{\mathcal{A},t}$ only. In particular, such a response will give the manager a continuation payoff of $v_t(h^{\mathcal{A},t})$ for any history $\tilde{h}^{\mathcal{M},t}$ that satisfies $h^{\mathcal{A},t} \subset \tilde{h}^{\mathcal{M},t}$. Consequently, $v_t(h^{\mathcal{M},t}) = v_t(h^{\mathcal{A},t})$.

Step 2b. We now show that if $v^* < v_0$, the only equilibrium is one in which all players take their outside options; and if $v^* \geq v_0$, then v^* bounds the manager's continuation payoffs from above.

Consider an arbitrary equilibrium and the agent's continuation payoff $u_t(h^{\mathcal{A},t})$. In the equilibrium, for each $\tau > t$, denote $w_\tau := \Pr[e_\tau = 1 \mid h^{\mathcal{A},t}]$ the agent's period t-belief that he will exert effort at period τ, and as $p_\tau := \Pr[\iota_\tau = 1 \mid (\omega_\tau, s_\tau) \neq (C, 0), e_\tau = 1, h^{\mathcal{A},t}]$ and $q_\tau := \Pr[\iota_\tau = 1 \mid (\omega_\tau, s_\tau) = (C, 0), e_\tau = 1, h^{\mathcal{A},t}]$ the agent's belief about implementation conditional on $e_\tau = 1$, ω_τ, s_τ, and his history $h^{\mathcal{A},t}$. Decompose $u_t(h^{\mathcal{A},t})$ as

$$
\begin{aligned}
u_t(h^{\mathcal{A},t}) &= (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\Pr((\omega_\tau, s_\tau) \neq (C,0)) p_\tau b + \Pr((\omega_\tau, s_\tau,) = (C,0)) q_\tau b - c\right] \\
&= (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\left(1 - \left(1 - \frac{1}{2}\sigma\right)\alpha\right) p_\tau b + \left(1 - \frac{1}{2}\sigma\right) \alpha q_\tau b - c\right].
\end{aligned} \tag{3}
$$

Similarly, decompose u^* as

$$
u^* = (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\left(1 - \left(1 - \frac{1}{2}\sigma\right)\alpha\right) b + \left(1 - \frac{1}{2}\sigma\right) \alpha \eta^* b - c\right], \tag{4}
$$

Since $b > 0$, $p_\tau \leq 1$ for all τ, and $u_t(h^{\mathcal{A},t}) \geq u^* = 0$ (recall that $u^* = 0$ follows from the definition of η^*), using (3) and (4) we obtain

$$
\sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau q_\tau \geq \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \eta^*. \tag{5}
$$

Step 2c. Note that in this equilibrium,

$$
\begin{aligned}
v_t(h^{\mathcal{A},t}) = v_0 + (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \Bigg[&\Pr((\omega_\tau, s_\tau) \neq (C,0)) p_\tau (y - v_0) + \\
&\Pr((\omega_\tau, s_\tau) = (C,0)) \alpha q_\tau (\mathbb{E}(V_{\mathcal{M}} \mid s = 0) y - v_0)\Bigg] \\
= v_0 + (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \Bigg[&\left(1 - \left(1 - \frac{1}{2}\sigma\right)\alpha\right) p_\tau (y - v_0) + \\
&\left(1 - \frac{1}{2}\sigma\right) \alpha q_\tau \left(\frac{1-\sigma}{2-\sigma} y - v_0\right)\Bigg].
\end{aligned}
$$

Using (5) and the fact that $y - v_0 > 0$, $p_\tau \leq 1$, and $\frac{1-\sigma}{2-\sigma} y - v_0 < 0$, it follows that

$$v_t(h^{\mathcal{A},t}) \leq v_0 + (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau k,$$

where

$$k := \left(1 - \left(1 - \frac{1}{2}\sigma\right)\alpha\right)(y - v_0) + \left(1 - \frac{1}{2}\sigma\right)\alpha\eta^* \left(\frac{1-\sigma}{2-\sigma} y - v_0\right).$$

Also note that v^* can be written as

$$v^* = v_0 + (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} k.$$

Now, $v^* < v_0 \Rightarrow k < 0$. Thus, if for any t the manager's continuation payoff $v_t(h^{\mathcal{M},t}) = v_t(h^{\mathcal{A},t}) \geq v_0$ we must have $w_\tau = 0$ for all $\tau > t$. That is, the agent never exerts effort in any future periods, and hence, the equilibrium is the one in which all players take their outside options (and $v_t = v_0$ for all t). On the other hand, $v^* \geq v_0 \implies k \geq 0$. In this case, as $w_\tau \leq 1$, $v^* \geq v_t(h^{\mathcal{A},t}) = v_t(h^{\mathcal{M},t})$.

Step 3. *Proof of second result in Part (iii)*: Finally, we show that the principal's continuation payoff is bounded above by π^* for any equilibrium. Recall that this continuation payoff is written as $\pi_t(h^{\mathcal{P},t})$ for any period t.

Consider an arbitrary equilibrium. For each $\tau > t$, with a slight abuse of notation, let $p_\tau := \Pr[\iota_\tau = 1 \mid s_\tau = 1, e_\tau = 1, h^{\mathcal{A},t}]$ and $q_\tau := \Pr[\iota_\tau = 1 \mid s_\tau = 0, e_\tau = 1, h^{\mathcal{A},t}]$ the agent's belief of implementation conditional on $e_\tau = 1$ and s_τ. (As before, we continue to denote $w_\tau := \Pr[e_\tau = 1 \mid h^{\mathcal{A},t}]$ as the agent's period t-belief that he will exert effort at period τ).

Step 3a. We can write

$$u_t(h^{\mathcal{A},t}) = (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\frac{1}{2}\sigma p_\tau b + \left(1 - \frac{1}{2}\sigma\right) q_\tau b - c\right],$$

and

$$u^* = (1-\delta) \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\frac{1}{2}\sigma b + \left(1 - \frac{1}{2}\sigma\right)(1 - \alpha(1-\eta^*))b - c\right].$$

Since $u_t \geq u^* = 0$, $b > 0$, and $p_\tau \leq 1$, we obtain

$$\sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau q_\tau \geq \sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau (1 - \alpha(1-\eta^*)). \tag{6}$$

Step 3b. Note that

$$\pi_t(h^{\mathcal{A},t}) = \pi_0 + (1-\delta)\sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \left[\frac{1}{2}\sigma p_\tau (Y-\pi_0) + \left(1-\frac{1}{2}\sigma\right) q_\tau \left(\frac{1-\sigma}{2-\sigma}Y - \pi_0\right)\right]$$
$$\leq \pi_0 + (1-\delta)\sum_{\tau \geq t+1} \delta^{\tau-t-1} w_\tau \Bigg[\frac{1}{2}\sigma(Y-\pi_0) +$$
$$\left(1-\frac{1}{2}\sigma\right)(1-\alpha(1-\eta^*))\left(\frac{1-\sigma}{2-\sigma}Y-\pi_0\right)\Bigg].$$

where the inequality follows from (6), and the facts that $\frac{1-\sigma}{2-\sigma}Y < \pi_0$, $Y - \pi_0 > 0$, and $p_\tau \leq 1$. Also note that π^* can be written as

$$\pi^* = \pi_0 + (1-\delta)\sum_{\tau \geq t+1} \delta^{\tau-t-1} \left[\frac{1}{2}\sigma(Y-\pi_0) + \left(1-\frac{1}{2}\sigma\right)(1-\alpha(1-\eta^*))\left(\frac{1-\sigma}{2-\sigma}Y-\pi_0\right)\right].$$

Since $\pi^* \geq \pi_0$,

$$\frac{1}{2}\sigma(Y-\pi_0) + \left(1-\frac{1}{2}\sigma\right)(1-\alpha(1-\eta^*))\left(\frac{1-\sigma}{2-\sigma}Y-\pi_0\right) > 0.$$

As $w_\tau \leq 1$, we obtain $\pi_t(h^{\mathcal{A},t}) \leq \pi^*$. Furthermore, because the principal's history is contained in that of the agent's, by the law of iterated expectation,

$$\mathbb{E}\left[\pi_t(h^{\mathcal{A},t}) \mid h^{\mathcal{P},t}\right] = \pi_t(h^{\mathcal{P},t}).$$

Thus, we obtain $\pi_t(h^{\mathcal{P},t}) \leq \pi^*$. □

Proof of Proposition 2. We begin by showing that the incentive compatibility constraint of the principal, ($IC_{\mathcal{P}}^M$) is always satisfied. Recall that this constraint requires that

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid r = 1)Y + \delta\pi^* \geq \pi_0.$$

Decomposing π^* based on the manager's report, we obtain

$$\pi^* = \Pr(r=1)\Pr(V_{\mathcal{P}} = Y \mid r=1)Y + \Pr(r=0)\pi_0.$$

Since $\pi^* \geq \pi_0$ (by Lemma 1), this implies that $\Pr(V_{\mathcal{P}} = Y \mid r = 1)Y \geq \pi_0$, which in turn implies that ($IC_{\mathcal{P}}^M$) is satisfied.

We now analyze the conditions under which the manager's incentive compatibility constraint

($IC^M_{\mathcal{M}}$) is satisfied. From (1), we obtain that

$$\eta^* = 1 - \frac{1-\frac{c}{b}}{\alpha\left(1-\frac{1}{2}\sigma\right)}.$$

Furthermore, observe that we can write

$$\begin{aligned} v^* &= \left[\frac{1}{2}\sigma + \left(1-\frac{1}{2}\sigma\right)(1-\alpha)\right] y + \frac{1}{2}(1-\sigma)\alpha\eta^* y + \left(1-\frac{1}{2}\sigma\right)\alpha(1-\eta^*)v_0 \\ &= \left(1-\frac{c}{b}\right)v_0 + \left(\frac{1}{2-\sigma} + \frac{c}{b}\left(\frac{1-\sigma}{2-\sigma}\right) - \frac{1}{2}\alpha\right) y, \end{aligned}$$

where the second equality follows from replacing η^* with its expression given above and simplifying.

In what follows, let $L_M(\alpha,\sigma,\delta)$ denote the left-hand side of ($IC^M_{\mathcal{M}}$). Differentiating L_M with respect to α, we obtain

$$\frac{\partial L_M}{\partial \alpha} = \delta \frac{\partial v^*}{\partial \alpha} = -\frac{1}{2}\delta y < 0.$$

Thus, L_M decreases with α. Now, note that when $(\alpha,\sigma) \in B$, $\alpha \geq \alpha_1(\sigma)$ and $\sigma \geq \sigma_0$. So, for any given $\sigma \geq \sigma_0$, the maximum value of L_M is obtained when $\alpha = \alpha_1(\sigma)$, and is given by

$$\begin{aligned} L_M(\alpha_1(\sigma),\sigma,\delta) &= (1-\delta)\left(\frac{1-\sigma}{2-\sigma}\right) y + \delta v^*\big|_{\alpha=\alpha_1(\sigma)} \\ &= (1-\delta)\left(\frac{1-\sigma}{2-\sigma}\right) y + \delta\left[\frac{c}{b}y + \left(1-\frac{c}{b}\right)v_0\right], \end{aligned}$$

where the second equality follows from using (1) and the expression of v^* above. Since $(1-\sigma)/(2-\sigma)$ decreases with σ, it is clear that L_M is maximized in B when $\alpha = \alpha_1(\sigma_0)$ and $\sigma = \sigma_0$. Thus, the threshold value of δ above which a mediation contract can be sustained, δ_M, is given by the equation $L_M(\alpha_1(\sigma_0),\sigma_0,\delta_M) = v_0$ or, equivalently,

$$(1-\delta_M)\left(\frac{1-\sigma_0}{2-\sigma_0}\right) y + \delta_M\left[\frac{c}{b}y + \left(1-\frac{c}{b}\right)v_0\right] = v_0.$$

Solving this equation with respect to δ_M, we obtain

$$\delta_M = \frac{1}{1+\frac{c}{b}S_M}, \tag{7}$$

where $S_M = ((y/v_0)-1)/\left(1-\left(\frac{1-\sigma_0}{2-\sigma_0}\right)(y/v_0)\right)$.

For any $\delta \geq \delta_M$ and $\sigma \geq \sigma_0$, define $\alpha_M(\sigma,\delta)$ as the value of α for which the constraint ($IC^M_{\mathcal{M}}$) holds with equality. Since L_M decreases with α, we immediately obtain that ($IC^M_{\mathcal{M}}$) is satisfied if and only if $\alpha \leq \alpha_M(\sigma,\delta)$. We next show that $\alpha_M(\sigma,\delta)$ increases with δ, but need not be monotone in σ. Using the Implicit Function theorem, we obtain

$$\frac{\partial}{\partial\delta}\alpha_M(\sigma,\delta) = -\frac{\partial L_M/\partial\delta}{\partial L_M/\partial\alpha} = \frac{2}{\delta y}\left(v^* - \frac{1-\sigma}{2-\sigma}y\right) > 0 \text{ and}$$
$$\frac{\partial}{\partial\sigma}\alpha_M(\sigma,\delta) = -\frac{\partial L_M/\partial\sigma}{\partial L_M/\partial\alpha} = \frac{2}{\delta y}\left[-(1-\delta)\frac{y}{(2-\sigma)^2} + \delta\frac{1-\frac{c}{b}}{(2-\sigma)^2}y\right].$$

Thus, $\alpha_M(\sigma,\delta)$ increases with δ, and it increases with σ if and only if $\delta > \frac{b}{2b-c} =: \hat{\delta}$. Whether $\delta_M \leq \hat{\delta}$ depends on whether

$$L_M(\alpha_1(\sigma_0),\sigma_0,\hat{\delta}) = (1-\hat{\delta})\left(\frac{1-\sigma_0}{2-\sigma_0}\right)y + \hat{\delta}\left[\frac{c}{b}y + \left(1-\frac{c}{b}\right)v_0\right] \geq v_0,$$

which is equivalent to

$$\frac{1}{2} + \frac{c}{b}\left(1-\frac{\pi_0}{Y}\right) \geq \frac{v_0}{y}.$$

Hence, when this condition is satisfied, $\alpha_M(\sigma,\delta)$ decreases in σ if $\delta \in [\delta_M, \hat{\delta}]$ and increases in σ if $\delta \in [\hat{\delta}, 1)$. And when this condition is violated, $\alpha_M(\sigma,\delta)$ increases in σ for all $\delta \in [\delta_M, 1)$. □

Proof of Proposition 3. As argued in the text, the intervention contract can be sustained if and only if the constraints ($IC^I_{\mathcal{M}}$) and ($IC^I_{\mathcal{P}}$) are satisfied.

Consider first the constraint ($IC^I_{\mathcal{M}}$). As shown in the proof of Proposition 2, we can write

$$v^* = \left(1-\frac{c}{b}\right)v_0 + \left(\frac{1}{2-\sigma} + \frac{c}{b}\left(\frac{1-\sigma}{2-\sigma}\right) - \frac{1}{2}\alpha\right)y. \tag{8}$$

Thus, $v^* \geq v_0$ if and only if

$$\alpha \leq \alpha_I(\sigma) := \frac{2}{2-\sigma} - \frac{2c}{b}\left(\frac{v_0}{y} - \frac{1-\sigma}{2-\sigma}\right),$$

meaning that ($IC^I_{\mathcal{M}}$) holds if and only if $\alpha \leq \alpha_I(\sigma)$. Since $\partial\alpha_I(\sigma)/\partial\sigma = 2(1-\frac{c}{b})/(2-\sigma)^2$ and by assumption $b > c$, we obtain that $\alpha_I(\sigma)$ increases in σ.

Consider now the constraint ($IC^I_{\mathcal{P}}$). Recall that this constraint requires that

$$(1-\delta)\frac{1}{2}Y + \delta\pi^* \geq \pi_0.$$

From (2) and the fact that

$$\eta^* = 1 - \frac{1-\frac{c}{b}}{\alpha\left(1-\frac{1}{2}\sigma\right)},$$

we obtain that

$$\pi^* = \left(\frac{1}{2}\sigma\left(\frac{1}{2-\sigma}\right) + \frac{c}{b}\left(\frac{1-\sigma}{2-\sigma}\right)\right)Y + \left(1-\frac{c}{b}\right)\pi_0, \tag{9}$$

which is independent of α. Thus, the left-hand side of ($IC^I_{\mathcal{P}}$) is also independent of α, and, in what follows, we denote it by $L_I(\sigma, \delta)$. Observe that

$$\frac{\partial L_I}{\partial \sigma} = \frac{\delta Y}{(2-\sigma)^2}\left(1 - \frac{c}{b}\right) > 0.$$

Hence, L_I increases in σ. Therefore, it is maximized in B when $\sigma = \sigma_1$ (which corresponds to point $(\alpha, \sigma) = (1, \sigma_1) = (1, \frac{2c}{b})$ in B). Moreover, note that ($IC^I_{\mathcal{M}}$) is satisfied when $(\alpha, \sigma) = (1, \sigma_1)$ since $v^* \mid_{(\alpha,\sigma)=(1,\frac{2c}{b})} = \frac{c}{b}y + (1 - \frac{c}{b})v_0 > v_0$ (i.e. $\alpha = 1 \leq \alpha_I(\sigma_1)$). Thus, the threshold value of δ above which the intervention contract be sustained, δ_I, is given by the equation $L_I(\sigma_1, \delta_I) = \pi_0$ or, equivalently,

$$(1-\delta_I)\frac{1}{2}Y + \delta_I \pi^* \big|_{\sigma\,=\,\sigma_1} = \pi_0.$$

Solving this equation (using (9)), we obtain

$$\delta_I = \frac{1}{1 + \frac{c}{b}S_I},$$

where $S_I = (Y/\pi_0 - 1)/\left(1 - \frac{1}{2}Y/\pi_0\right)$. For any $\delta \geq \delta_I$ (and $\sigma \geq \sigma_0$), define $\sigma_I(\delta)$ as the value of σ for which ($IC^I_{\mathcal{P}}$) holds with equality. Since L_I increases in σ, ($IC^I_{\mathcal{P}}$) is satisfied if and only if $\sigma \geq \sigma_I(\delta)$. Moreover,

$$\frac{\partial}{\partial \delta}\sigma_I(\delta) = -\frac{\partial L_I/\partial \delta}{\partial L_I/\partial \sigma} = -\frac{(2-\sigma)^2}{\delta Y(1-\frac{c}{b})}\left(\pi^* - \frac{1}{2}Y\right).$$

By Lemma 1, $\pi^* \geq \pi_0 > \frac{1}{2}Y$. Hence, $\partial \sigma_I(\delta)/\partial \delta < 0$, implying that $\sigma_I(\delta)$ decreases in δ. Finally, it follows from the definition of δ_I and of σ_I that $\sigma_I(\delta_I) = \sigma_1 = \frac{2c}{b}$. And note that when $\delta = 1$, ($IC^I_{\mathcal{P}}$) reduces to $\pi^* \geq \pi_0$, implying that $\sigma_I(1) = \sigma_0$. □

Proof of Lemma 2. **Step 1.** If $v^* < v_0$, then by Lemma 1 the equilibrium is unique, and in this equilibrium all players take their outside options in all periods. So, in this case, the claim is trivial.

Step 2. Next, consider the case where $v^* \geq v_0$. We prove the result by contradiction. Suppose there exist equilibria in which, in some period(s), all players participate and the agent exerts effort. Consider one such equilibrium, arbitrarily chosen.

Step 2a. First, we claim that there cannot exists any history $h^{\mathcal{P},t}$ on the equilibrium path such that, given $h^{\mathcal{P},t}$, the manager babbles and the principal implements the project (regardless of the manager's recommendation). To see this, note that implementing the project in such circumstances would have to be incentive compatible for the principal:

$$(1-\delta)\frac{1}{2}Y + \delta\breve{\pi} \geq \pi_0,$$

where $\breve{\pi} := \mathbb{E}[\pi_{t+1}(h^{\mathcal{P},t+1}) \mid h^{\mathcal{P},t}, d_t, e_t, x_t, r_t, \iota_t = 1]$ denotes the principal's expected payoff from the continuation game. However, this condition cannot be satisfied because $\breve{\pi} \leq \pi^*$ (by Lemma 1, $\pi_{t+1}(h^{\mathcal{P},t+1}) \leq \pi^*$ for each $h^{\mathcal{P},t+1}$ on the equilibrium path) and ($IC^I_{\mathcal{P}}$) is violated (as we are assuming that the intervention contract cannot be sustained in equilibrium).

Step 2b. Next, we claim that for there cannot exists any history $h^{\mathcal{P},t}$ on the equilibrium path such that, given $h^{\mathcal{P},t}$, the implementation is contingent on the manager's recommendation ($\iota_t = 1$ if and only if $r_t = 1$), and the manager may recommend the project if $(\omega_t, s_t) = (C, 0)$. To see this, observe that recommending the project would have to be incentive compatible for the manager:

$$(1-\delta)\frac{1-\sigma}{2-\sigma}y + \delta\breve{v} \geq v_0,$$

where $\breve{v} := \mathbb{E}[v_{t+1}(h^{\mathcal{M},t+1}) \mid h^{\mathcal{M},t}, d_t, e_t, x_t, z_t, r_t = 1]$ denotes the manager's expected payoff from the continuation game. However, this condition cannot be satisfied because $\breve{v} \leq v^*$ (by Lemma 1, $v_{t+1}(h^{\mathcal{P},t+1}) \leq v^*$ for each $h^{\mathcal{M},t+1}$ on the equilibrium path) and ($IC^M_{\mathcal{M}}$) is violated (as we are assuming a mediation contract cannot be sustained in equilibrium).

Step 2c. Now, note that the two situations considered above are the only ones in which a project can get implemented if $(\omega, s) = (C, 0)$ (recall that we focus on equilibria in pure strategies). Since none of them are feasible, in equilibrium, no project with $(\omega, s) = (C, 0)$ is implemented.

Step 2d. Denote $w_\tau := \Pr[e_\tau = 1 \mid h^{\mathcal{A},\tau}]$, $p_\tau := \Pr[\iota_\tau = 1 \mid (\omega_\tau, s_\tau) \neq (C,0), e_\tau = 1, h^{\mathcal{A},t}]$, and $q_\tau := \Pr[\iota_\tau = 1 \mid (\omega_\tau, s_\tau) = (C,0), e_\tau = 1, h^{\mathcal{A},t}] = 0$. Then, consider the decomposition of the agent's payoff u at the beginning of period 1:

$$\begin{aligned}
u &= (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\Pr((\omega_\tau, s_\tau,) \neq (C,0))p_\tau b + \Pr((\omega_\tau, s_\tau,) = (C,0))q_\tau b - c\right]\\
&= (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\Pr((\omega_\tau, s_\tau,) \neq (C,0))p_\tau b - c\right]\\
&\leq (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\Pr((\omega_\tau, s_\tau,) \neq (C,0))b - c\right].
\end{aligned}$$

As $u \geq 0$ in any equilibrium, and $\Pr((\omega_\tau, s_\tau,) \neq (C,0))b < c$ for $(\alpha, \sigma) \in B$, we must have $w_\tau = 0$ for all $\tau \geq 1$. Therefore, in the equilibrium we consider, $e_\tau = 0$ for all $\tau \geq 1$, which contradicts with the assumption that the agent may exert effort in some period(s). □

Proof of Theorem 1. Fix $\delta \in [\delta_I, 1)$. Since $\delta \geq \delta_I$, we know by Proposition 3 that $B_I \neq \emptyset$. We next analyze B_M and how it compares to B_I.

Step 1. By Proposition 2, we know that $B_M \neq \emptyset$ if and only if $\delta \geq \delta_M$. From (7) in the proof of Proposition 2, we obtain that δ_M is strictly decreasing in y/v_0, $\delta_M \to 1$ as $y/v_0 \to 1$, and $\delta_M \to 0$ as $y/v_0 \to (2-\sigma_0)/(1-\sigma_0)$. Since δ_M is continuous and decreasing in y/v_0, there exists $\tau_1 \in (1, (2-\sigma_0)/(1-\sigma_0))$ such that $\delta_M = \delta$ if $y/v_0 = \tau_1$, and $\delta_M \leq \delta$ if and only if $y/v_0 \geq \tau_1$. Hence, $B_M = \emptyset$ if $y/v_0 < \tau_1$. This establishes part (i) of the proposition.

Step 2. We now prove part (iii) of the proposition. Consider the point $(\alpha, \sigma) = (1, \sigma_I(\delta)) \in B$. If it is also the case that $(1, \sigma_I(\delta)) \in B_M$, then by Proposition 3 and Proposition 4, we obtain that $B_I \subseteq B_M$. Now, observe that $(1, \sigma_I(\delta)) \in B_M$ if only if ($IC^M_{\mathcal{M}}$) is satisfied at this point, i.e. if and only if

$$(1-\delta)\left(\frac{1-\sigma_I}{2-\sigma_I}\right) y + \delta v^*|_{(\alpha,\sigma)\,=\,(1,\sigma_I)} \geq v_0. \tag{10}$$

Plugging (8) in this inequality and rearranging, we obtain

$$\frac{y}{v_0} \geq \frac{(2-\sigma_I)\left[1-\delta(1-\frac{c}{b})\right]}{(1-\delta)(1-\sigma_I)+\frac{1}{2}\delta\sigma_I+\delta\frac{c}{b}(1-\sigma_I)} =: \tau_2.$$

Thus, if $y/v_0 \geq \tau_2$, then $B_I \subseteq B_M$. Furthermore, observe that by definition of σ_I, ($IC^I_{\mathcal{P}}$) holds with equality at the point $(\alpha, \sigma) = (1, \sigma_I(\delta))$, implying that

$$(1-\delta)\frac{1}{2}Y + \delta\pi^*|_{(\alpha,\sigma)=(1,\sigma_I)} = \pi_0.$$

Plugging (2) in this equation and rearranging, we obtain

$$\frac{Y}{\pi_0} = \frac{(2-\sigma_I)\left[1-\delta(1-\frac{c}{b})\right]}{(1-\delta)(1-\sigma_I)+\frac{1}{2}\sigma_I+\delta\frac{c}{b}(1-\sigma_I)}.$$

It is clear from direct of inspection of this equation and the expression for τ_2 derived above that $\tau_2 > Y/\pi_0$.

Step 3. We now argue that $\tau_2 > \tau_1$.

Step 3a. Recall that whenever $y/v_0 \geq \tau_2$, we have $(1, \sigma_I(\delta)) \in B_M$. Then, as $B_M \neq \emptyset$, $y/v_0 \geq \tau_1$. Since $y/v_0 \geq \tau_1$ for any ratio $y/v_0 \geq \tau_2$, we conclude $\tau_2 \geq \tau_1$.

Step 3b. Next, we rule out the case that $\tau_2 = \tau_1$. We prove this by contradiction. Suppose that $\tau_2 = \tau_1$, and let $y/v_0 = \tau_1 = \tau_2$. Then, by the definition of τ_1 and the definition of δ_M (see the proof of Proposition 2), we know that $\delta = \delta_M$, and

$$(1-\delta)\left(\frac{1-\sigma_0}{2-\sigma_0}\right) y + \delta v^*|_{(\alpha,\sigma)\,=\,(\alpha_1(\sigma_0),\sigma_0)} = v_0.$$

Since $\sigma_I(\delta) > \sigma_0$ (see Proposition 3),

$$\frac{1-\sigma_0}{2-\sigma_0} > \frac{1-\sigma_I(\delta)}{2-\sigma_I(\delta)}.$$

In addition, the proof of Proposition 2 has shown that

$$v^*|_{(\alpha,\sigma)\,=\,(\alpha_1(\sigma_0),\sigma_0)} > v^*|_{(\alpha,\sigma)\,=\,(1,\sigma_I(\delta))}.$$

Combining these two facts, we have

$$(1-\delta)\frac{1-\sigma_I(\delta)}{2-\sigma_I(\delta)}y+\delta v^*|_{(\alpha,\sigma)\,=\,(1,\sigma_I(\delta))}<v_0,$$

implying that (10) fails and $y/v_0<\tau_2$. This is a contradiction.

Step 4. Finally, we prove part (ii) of the proposition. That is, we show that if $y/v_0\in[\tau_1,\tau_2)$, then $B_M\backslash B_I\neq\emptyset$ and $B_I\backslash B_M\neq\emptyset$.

Step 4a. Since $y/v_0\geq\tau_1$, then $B_M\neq\emptyset$, which implies that $(\alpha_1(\sigma_0),\sigma_0)\in B_M$. However, because $\sigma_I(\delta)>\sigma_0$, we obtain that $(\alpha_1(\sigma_0),\sigma_0)\notin B_I$ and, therefore, $B_M\backslash B_I\neq\emptyset$.

Step 4b. To show that $B_I\backslash B_M\neq\emptyset$, we consider two possible cases separately: (i) the case where the constraint $\alpha\leq\overline{\alpha}(\sigma)$ is satisfied at $(\alpha,\sigma)=(1,\sigma_I(\delta))$ and (ii) the case when it is not. Regarding the first case, since $\alpha\leq\overline{\alpha}(\sigma)$ is satisfied at $(\alpha,\sigma)=(1,\sigma_I(\delta))$, then $(1,\sigma_I(\delta))\in B_I$; and because $y/v_0<\tau_2$, $(1,\sigma_I(\delta))\notin B_M$. Thus, $B_I\backslash B_M\neq\emptyset$.

Consider now the second case where $\alpha\leq\overline{\alpha}(\sigma)$ is not satisfied at $(\alpha,\sigma)=(1,\sigma_I(\delta))$. Because $\alpha\leq\overline{\alpha}(\sigma)$ is violated at $(\alpha,\sigma)=(1,\sigma_I(\delta))$, this point does not satisfy ($IC^I_{\mathcal{M}}$), implying that $(1,\sigma_I(\delta))\notin B_I$. However, the point $(\alpha,\sigma)=(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))$ where ($IC^I_{\mathcal{M}}$) just holds, satisfies both ($IC^I_{\mathcal{M}}$) and ($IC^I_{\mathcal{P}}$) and, therefore, is in B_I. Now, since ($IC^I_{\mathcal{M}}$) holds with equality at $(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))$, we know that $v^*=v_0$ at this point. Because $v^*=v_0$ and $\frac{1-\sigma_I}{2-\sigma_I}y\leq\frac{1}{2}y<v_0$, we obtain that ($IC^M_{\mathcal{M}}$) is necessarily violated at $(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))$, and $(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))\notin B_M$. Since $(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))\in B_I$ and $(\overline{\alpha}(\sigma_I(\delta)),\sigma_I(\delta))\notin B_M$, $B_I\backslash B_M\neq\emptyset$. □

Proof of Lemma 3*.* That δ_M is decreasing in y/v_0 follows directly from (7) and the observation that S_M is increasing in y/v_0.

Next, note that, by definition, at α_M ($IC^M_{\mathcal{M}}$) binds. Thus, we have

$$\tilde{L}_M:=(1-\delta)\left(\frac{1-\sigma}{2-\sigma}\right)\frac{y}{v_0}+\delta\frac{v^*}{v_0}\bigg|_{\alpha\,=\alpha_M}\equiv 1.$$

Also, from the expression for v^* as given in the proof of Proposition 2), it readily follows that $\partial(v^*/v_0)/\partial(y/v_0)>0$. Thus, using the Implicit Function theorem, we obtain

$$\frac{\partial\alpha_M}{\partial(y/v_0)}=-\frac{\partial\tilde{L}_M/\partial(y/v_0)}{\partial\tilde{L}_M/\partial\alpha}=\frac{2}{\delta y}\left((1-\delta)\left(\frac{1-\sigma}{2-\sigma}\right)+\delta\left(\frac{\partial(v^*/v_0)}{\partial(y/v_0)}\right)\right)>0.$$

□

Proof of Corollary 1*.* Suppose that $\{z_t\}_{t\geq1}$ is public and fix some $(\alpha,\sigma)\in B\backslash B_{\mathcal{I}}$. Consider an arbitrary equilibrium where at least in one period all players participate and the agent exerts effort.

Step 1. We first prove that in this equilibrium, there exists a period t and a realization of (x_t,z_t) such that the manager recommends the project even if $(\omega_t,s_t)=(C,0)$.

Suppose not. Then, from Step 2d in the proof of Lemma 2, we know that the agent's payoff u at the beginning of period 1 is

$$u \leq (1-\delta) \sum_{\tau \geq 1} \delta^{\tau-1} w_\tau \left[\Pr((\omega_\tau, s_\tau,) \neq (C, 0)) b - c \right] < 0.$$

But this is a contradiction as in any equilibrium, $u \geq 0$.

Step 2. As in equilibrium, the manager recommends the project when $(\omega_t, s_t) = (C, 0)$, it is optimal for him to also recommend the project when $(\omega_t, s_t) = (D, 0)$. Then, in the principal's incentive constraint

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid r = 1, x_t, z_t) Y + \delta \pi^* \geq \pi_0,$$

we have $\Pr(V_{\mathcal{P}} = Y \mid r = 1, x_t, z_t) \leq \frac{1}{2}$. Therefore, ($IC^I_{\mathcal{P}}$) must be satisfied. Furthermore, as we assume that all players participate in at least one period, $IC^I_{\mathcal{M}}$ must hold as well. But this implies $(\alpha, \sigma) \in B_I$, and contradicts the fact that $(\alpha, \sigma) \in B \backslash B_{\mathcal{I}}$. □

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Online Appendix: "Managing Loyalty in Hierarchical Organizations"

Hanzhe Li, Jin Li, Arijit Mukherjee, and Luis Vasconcelos

In this appendix we present the proof of Proposition 4. To show this result, we first present several useful definitions and observations, then prove the proposition as a corollary.

Denote the payoff profile from the 0-lenient action as $(u_{l0}, v_{l0}, \pi_{l0})$ and that from the $1-$lenient action as $(u_{l1}, v_{l1}, \pi_{l1})$. Note that these two profiles can be written as

$$u_{l0} = \frac{1}{2}\sigma b - c,\ v_{l0} = \frac{1}{2}\sigma y + \left(1 - \frac{1}{2}\sigma\right) v_0,\ \pi_{l0} = \frac{1}{2}\sigma Y + \left(1 - \frac{1}{2}\sigma\right) \pi_0,$$

and

$$u_{l1} = \left[1 - \alpha\left(1 - \frac{1}{2}\sigma\right)\right] b - c,\ v_{l1} = \frac{1}{2}\sigma y + (1-\alpha)\left(1 - \frac{1}{2}\sigma\right) y + \alpha\left(1 - \frac{1}{2}\sigma\right) v_0,$$
$$\pi_{l1} = \frac{1}{2}\sigma Y + \frac{1}{2}(1-\alpha)(1-\sigma) Y + \alpha\left(1 - \frac{1}{2}\sigma\right) \pi_0.$$

The payoff profile from a β-lenient action is given by

$$(u_{l\beta}, v_{l\beta}, \pi_{l\beta}) := \beta(u_{l1}, v_{l1}, \pi_{l1}) + (1-\beta)(u_{l0}, v_{l0}, \pi_{l0}).$$

Since $u_{l1} > u_{l0}$, $v_{l1} > v_{l0}$, and $\pi_{l1} < \pi_{l0}$, we know that with respect to the lenience level β, both $u_{l\beta}$ and $v_{l\beta}$ are strictly increasing, but $\pi_{l\beta}$ is strictly decreasing.

Define

$$\beta_* := \min\{\beta \in [0,1] : u_{l\beta} \geq 0\}.$$

That is, β_* is the lowest lenience level that brings to the agent a zero payoff. Similarly, define

$$\beta^* := \max\{\beta \in [0,1] : \pi_{l\beta} \geq \pi_0\}.$$

That is, β^* is the highest lenience level that brings to the principal a payoff of at least π_0. Denote the payoff profile from the β_*-lenient action as $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ and that from the β^*-lenient action as $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$. We observe the following about these actions and their payoffs (Figure 6). Note that in this lemma, the definitions of $\alpha_0(\sigma)$, $\alpha_1(\sigma)$, σ_0, and σ_1 remain the same as those given in Proposition 1.

Lemma 4. *Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma \geq \sigma_0$. The following hold:*

(i) If $\alpha < \alpha_0(\sigma)$ and $\sigma \geq \sigma_1$, then $\beta_* = 0$ and $0 < \beta^* < 1$.

(ii) If $\alpha < \alpha_0(\sigma)$ and $\sigma_0 < \sigma < \sigma_1$, then $0 < \beta_* < \beta^* < 1$.

(iii) If $\alpha \geq \alpha_0(\sigma)$ and $\sigma \geq \sigma_1$, then $\beta_* = 0$ and $\beta^* = 1$.

(iv) *If* $\alpha \geq \alpha_0(\sigma)$ *and* $\sigma_0 < \sigma < \sigma_1$*, then* $0 < \beta_* < 1$ *and* $\beta^* = 1$*.*

(v) *If* $\sigma = \sigma_0$*, then* $\beta_* = \beta^* \in (0, 1)$*.*

(vi) $v_{l\beta} > v_0$ *for any* $\beta \in [0, 1]$*.*

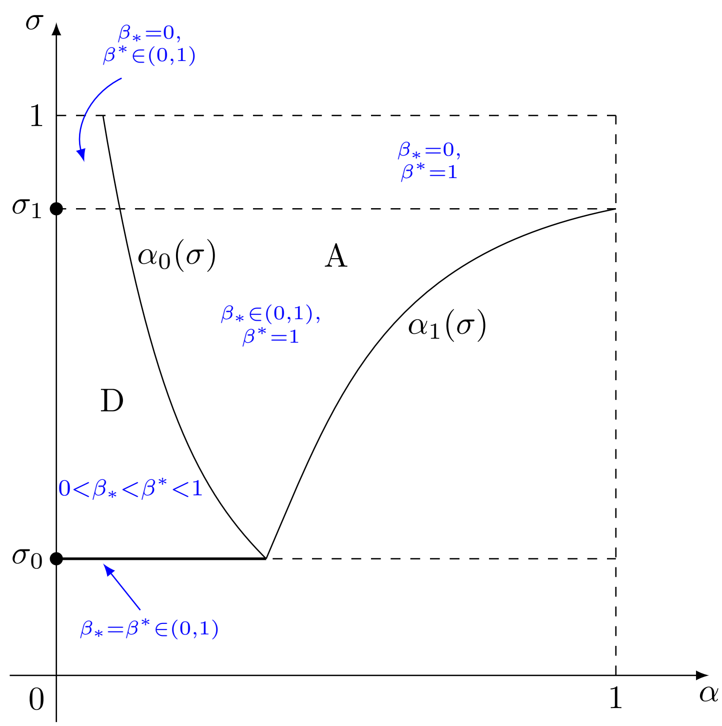

Figure 6: Illustration of Lemma 4

Proof. First, we prove the results concerning only β_*. Given the β_*-lenient action, any project with $s = 1$ is implemented with probability one, any project with $(\omega, s) = (D, 0)$ is implemented with probability β_*, and any project with $(\omega, s) = (C, 0)$ is never implemented. As the agent gains at least zero from the β_*-lenient action, we have

$$[\Pr(s = 1) + \Pr(\omega = D, s = 0)\beta_*]b = \left[\frac{1}{2}\sigma + (1 - \alpha)(1 - \frac{1}{2}\sigma)\beta_*\right] b \geq c.$$

From this inequality we can solve that

$$\beta_* = \begin{cases} 0, & \text{if } \sigma \geq \sigma_1 = \frac{2b}{c} \\ \frac{1}{2-\sigma}\left[\frac{2c}{b} - \sigma\right] > 0, & \text{if } \sigma < \sigma_1. \end{cases} \tag{11}$$

Next, consider β^*. Proposition 1 shows that the 1-lenient action is a Nash equilibrium of the stage game if and only if $\alpha \geq \alpha_0(\sigma)$. Thus, if $\alpha \geq \alpha_0(\sigma)$, then $\beta^* = 1$; and if $\alpha < \alpha_0(\sigma)$, then $\beta^* < 1$.

To compare β_* and β^*, we first write out principal's payoff from a β-lenient action:

$$\begin{aligned}\pi_{l\beta} &= \beta\pi_{l1} + (1-\beta)\pi_{l0} \\ &= \frac{1}{2}\sigma Y + \frac{1}{2}(1-\alpha)(1-\sigma)\beta Y + (\alpha\beta + 1 - \beta)\left(1 - \frac{1}{2}\sigma\right)\pi_0.\end{aligned} \tag{12}$$

Because $\pi_{l1} < \pi_{l0}$, $\pi_{l\beta}$ is strictly decreasing in β. Further, from (11) and (12), it is straightforward to show that $\pi_{l\beta_*} = \pi_0$ when $\sigma = \sigma_0 := 2c(2\pi_0 - Y)/[2c(\pi_0 - Y) + Yb]$ and $\pi_{l\beta_*} > \pi_0$ when $\sigma > \sigma_0$. Since the β^*-lenient action gives the principal a payoff at least equal to π_0, we conclude that when $\sigma = \sigma_0$, $\beta^* = \beta_*$; and when $\sigma > \sigma_0$, $\beta^* > \beta_*$. In addition, because $\alpha_0(\sigma_0) = \alpha_1(\sigma_0)$, we know that when $\sigma = \sigma_0$ and $\alpha < \alpha_1(\sigma_0)$, $\beta_* = \beta^* < 1$.

Finally, note that both the 0-lenient and the 1-lenient actions yield the manager a payoff strictly greater than v_0. Hence, as a convex combinations of these actions, $v_{l\beta} > v_0$ for any $\beta \in [0, 1]$. □

The following lemma establishes $\pi_{l\beta_*}$ as an upper bound of the principal's equilibrium payoff.

Lemma 5. *Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma \geq \sigma_0$. Then, the principal's payoff is weakly smaller than $\pi_{l\beta_*}$ in any equilibrium.*

Proof. Consider an arbitrary equilibrium. Denote the principal's equilibrium payoff as π, and the agent's equilibrium payoff as u. In the equilibrium, for each $\tau \geq 1$, let $p_\tau := \Pr[\iota_\tau = 1 \mid s_\tau = 1, e_\tau = 1]$ and $q_\tau := \Pr[\iota_\tau = 1 \mid s_\tau = 0, e_\tau = 1]$ denote the implementation probability conditional on $e_\tau = 1$ and s_τ. Notice that these probabilities are evaluated at the beginning of the repeated game, where the principal and the agent share the same information. In addition, we denote $w_\tau := \Pr[e_\tau = 1]$ as the probability that the agent exerts effort at period τ.

First, suppose $\sigma < \sigma_1$. Then, according to Lemma 4, $\beta_* > 0$, implying $u_{l\beta_*} = 0$. We can express u and $u_{l\beta_*}$ as:

$$u = (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\frac{1}{2}\sigma p_\tau b + \left(1 - \frac{1}{2}\sigma\right)q_\tau b - c\right],$$

$$u_{l\beta_*} = (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\frac{1}{2}\sigma b + \left(1 - \frac{1}{2}\sigma\right)(1-\alpha)\beta_* b - c\right].$$

Since $u \geq 0 = u_{l\beta_*}$, $b > 0$, and $p_\tau \leq 1$, we obtain

$$\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau q_\tau \geq \sum_{\tau\geq 1}\delta^{\tau-1}w_\tau(1-\alpha)\beta_*. \tag{13}$$

Then, we have

$$\pi = \pi_0 + (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\frac{1}{2}\sigma p_\tau(Y - \pi_0) + \left(1 - \frac{1}{2}\sigma\right)q_\tau\left(\frac{1-\sigma}{2-\sigma}Y - \pi_0\right)\right] \tag{14}$$

$$\leq \pi_0 + (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}w_\tau\left[\frac{1}{2}\sigma(Y - \pi_0) + \left(1 - \frac{1}{2}\sigma\right)(1-\alpha)\beta_*\left(\frac{1-\sigma}{2-\sigma}Y - \pi_0\right)\right]. \tag{15}$$

where the inequality follows from (13), and the facts that $\frac{1-\sigma}{2-\sigma}Y < \pi_0$, $Y - \pi_0 > 0$, and $p_\tau \leq 1$.

Note that $\pi_{l\beta_*}$ can be written as

$$\pi_{l\beta_*} = \pi_0 + (1-\delta)\sum_{\tau\geq 1}\delta^{\tau-1}\left[\frac{1}{2}\sigma(Y-\pi_0) + \left(1-\frac{1}{2}\sigma\right)(1-\alpha)\beta_*\left(\frac{1-\sigma}{2-\sigma}Y - \pi_0\right)\right]. \tag{16}$$

Since $\pi_{l\beta_*} \geq \pi_{l\beta^*} \geq \pi_0$,

$$\frac{1}{2}\sigma(Y-\pi_0) + \left(1-\frac{1}{2}\sigma\right)(1-\alpha)\beta_*\left(\frac{1-\sigma}{2-\sigma}Y - \pi_0\right) \geq 0.$$

Then, as $w_\tau \leq 1$ in (15), we obtain $\pi \leq \pi_{l\beta_*}$.

Now, suppose $\sigma \geq \sigma_1$. In this case, $\beta_* = 0$ (Lemma 4). Since $Y - \pi_0 > 0$ and $\frac{1-\sigma}{2-\sigma}Y < \pi_0$, we can compare (14) and (16) to show that $\pi \leq \pi_{l\beta_*}$. □

Lemma 5 implies that in regions A and D, an equilibrium is automatically optimal for the principal if her payoff is $\pi_{l\beta_*}$. Recall that $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ is induced by the β_*-lenient action. The next lemma shows that when the players are sufficiently patient, $\pi_{l\beta_*}$ can be sustained as an equilibrium payoff by repeating the β_*-lenient action; and if the players are not sufficiently patient, $\pi_{l\beta_*}$ cannot be sustained by any stationary contract.

Lemma 6. *Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma \geq \sigma_0$. There exists $\bar{\delta} \in (0,1)$ (which depends on α and σ) such that if $\delta \geq \bar{\delta}$, the β_*-lenient action is self-enforcing. Furthermore, if $\delta < \bar{\delta}$, $\pi_{l\beta_*}$ cannot be sustained as an equilibrium payoff by any stationary contract.*

Proof. Suppose $\sigma \geq \sigma_0$. We first consider the case when $\alpha_0(\sigma) \leq \alpha < \alpha_1(\sigma)$. Let the players coordinate on the public randomizing device x_t: if $x_t \in [0, \beta_*]$, then the manager sends $r_t = 1$ if and only if $(\omega_t, s_t) \neq (C, 0)$, and if $x_t \in (\beta_*, 1]$, the manager sends $r_t = 1$ if and only if $s_t = 1$. The agent exerts effort, and the principal follows the manager's recommendation. If any player is publicly known to have deviated, the repeated game is terminated.

By the definition of β_* and Lemma 4, $u_{l\beta_*} \geq 0$, $v_{l\beta_*} > v_0$, and $\pi_{l\beta_*} \geq \pi_0$. Therefore, their participation constraints are satisfied. To enforce the β_*-lenient action, there are four incentive constraints:

$$(1-\delta)[-c + \Pr(s=1)b + \Pr(\omega = D, s=0)\beta_* b] + \delta u_{l\beta_*} \geq 0, \tag{$IC_{\mathcal{A}}^{\beta_*}$}$$

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid r=1, x \leq \beta_*)Y + \delta\pi_{l\beta_*} \geq \pi_0, \tag{$IC_{\mathcal{P},1}^{\beta_*}$}$$

$$(1-\delta)\Pr(V_{\mathcal{P}} = Y \mid r=1, x > \beta_*)Y + \delta\pi_{l\beta_*} \geq \pi_0, \text{ and} \tag{$IC_{\mathcal{P},2}^{\beta_*}$}$$

$$\begin{aligned}(1-\delta)v_0 + \delta v_{l\beta_*} &\geq (1-\delta)y + \delta\left[\Pr(V_{\mathcal{P}} = Y \mid s=0)v_{l\beta_*} + \Pr(V_{\mathcal{P}} = 0 \mid s=0)v_0\right] \\ &\geq (1-\delta)y + \delta\left[\frac{1-\sigma}{2-\sigma}v_{l\beta_*} + \frac{1}{2-\sigma}v_0\right].\end{aligned} \tag{$IC_{\mathcal{M}}^{\beta_*}$}$$

($IC_{\mathcal{A}}^{\beta_*}$) indicates that the agent is willing to exert effort. ($IC_{\mathcal{P},1}^{\beta_*}$) and ($IC_{\mathcal{P},2}^{\beta_*}$) indicate that the principal is willing to follow the manager's recommendation. Finally, ($IC_{\mathcal{M}}^{\beta_*}$) indicates that upon $x > \beta_*$, the manager is willing to send $r = 0$ when $(\omega, s) = (D, 0)$.

Observe that ($IC_{\mathcal{A}}^{\beta_*}$) is equivalent to $u_{l\beta_*} \geq 0$. Hence, it is satisfied. From ($IC_{\mathcal{P}}$) and $\alpha \geq \alpha_0(\sigma)$ we know that $\Pr(V_{\mathcal{P}} = Y \mid r = 1, x \leq \beta_*)Y \geq \pi_0$. Then, because $\pi_{l\beta_*} \geq \pi_0$, ($IC_{\mathcal{P},1}^{\beta_*}$) is satisfied. And since $\Pr(V_{\mathcal{P}} = Y \mid r = 1, x \leq \beta_*)Y \leq \Pr(V_{\mathcal{P}} = Y \mid r = 1, x > \beta_*)Y$, ($IC_{\mathcal{P},2}^{\beta_*}$) is also satisfied. For the manager's incentive constraint, observe that the difference between the left-hand side and the right-hand side is continuous and increasing in δ, which is strictly positive when $\delta = 1$ (recall that $v_{l\beta_*} > v_0$) and strictly negative when $\delta = 0$. Hence, there exists $\bar{\delta} \in (0, 1)$ such that ($IC_{\mathcal{M}}^{\beta_*}$) holds if and only if $\delta \geq \bar{\delta}$. This completes our proof for the case when $\alpha_0(\sigma) \leq \alpha < \alpha_1(\sigma)$.

Next, we consider the case when $\alpha < \alpha_0(\sigma)$. In this case, $\Pr(V_{\mathcal{P}} = Y \mid r = 1, x \leq \beta_*)Y < \pi_0$, so ($IC_{\mathcal{P},1}^{\beta_*}$) may be violated. However, we can resolve this issue with the private randomizing device z_t. Instead of x_t, let the players coordinate on z_t: if $z_t \in [0, \beta_*]$, then $r_t = 1 \iff (\omega_t, s_t) \neq (C, 0)$; if $z_t \in (\beta_*, 1]$, then $r_t = 1 \iff s_t = 1$; and if the manager is known to have deviated by the agent, they leave the game forever. As the principal does not observe z_t, her incentive constraint is simply

$$(1 - \delta) \Pr(V_{\mathcal{P}} = Y \mid r = 1, \beta_*)Y + \delta \pi_{l\beta_*} \geq \pi_0.$$

Then, since $\pi_{l\beta_*} \geq \pi_0$ and $\pi_{l\beta_*} = \Pr(r = 1) \Pr(V_{\mathcal{P}} = Y \mid r = 1, \beta_*)Y + \Pr(r = 0)\pi_0$, we have $\Pr(V_{\mathcal{P}} = Y \mid r = 1, \beta_*)Y \geq \pi_0$. Together with $\pi_{l\beta_*} \geq \pi_0$, this implies that the above incentive constraint is satisfied. With this fact established, the argument in the previous paragraph follows.

Finally, suppose $\delta < \bar{\delta}$. If $\pi_{l\beta_*}$ is sustained as an equilibrium payoff by some stationary contract, it is implied that the β_*-lenient action is self-enforcing. However, when $\delta < \bar{\delta}$, the β_*-lenient action cannot be self-enforcing because ($IC_{\mathcal{M}}^{\beta_*}$) is violated. □

Lemma 6 shows that there exists $\bar{\delta}$ such that $\pi_{l\beta_*}$ can be sustained as an equilibrium payoff by stationary contracts if and only if $\delta \geq \bar{\delta}$. The proof of the lemma reveals two typical methods to enforce the β_*-lenient action: one through coordination on x_t and the other on z_t. These two methods can be used to enforce actions at any lenience level. Thus, it is worthwhile to formally define them.

We refer to the following as *enforcing a β-action through coordination on x_t*: If $x_t \in [0, \beta]$, then the manager sends $r_t = 1$ if and only if $(\omega_t, s_t) \neq (C, 0)$, and if $x_t \in (\beta, 1]$, then the manager sends $r_t = 1$ if and only if $s_t = 1$. The agent exerts effort, and the principal follows the manager's recommendation. If any player is publicly known to have deviated, the repeated game is terminated.

Similarly, we refer to the following as *enforcing a β-action through coordination on z_t*: If $z_t \in [0, \beta]$, then the manager sends $r_t = 1$ if and only if $(\omega_t, s_t) \neq (C, 0)$, and if $z_t \in (\beta, 1]$, then the manager sends $r_t = 1$ if and only if $s_t = 1$. The agent exerts effort, and the principal follows the manager's recommendation. If any player is publicly known to have deviated, the repeated game is terminated. Additionally, if the agent observes the manager's deviation, both the manager and the agent leave the game forever.

Using these methods, the next lemma shows that the β^*-lenient action can also be self-enforcing as long as the players are sufficiently patient. For $\sigma > \sigma_0$, the required patience level is strictly lower than $\bar{\delta}$.

Lemma 7. *Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma > \sigma_0$. There exists $\underline{\delta}_1 \in (0, \bar{\delta})$ (which depends on α and σ) such that if $\delta \geq \underline{\delta}_1$, the β^*-lenient action is self-enforcing.*

Proof. First, consider the case when $\alpha_0(\sigma) \leq \alpha < \alpha_1(\sigma)$. In this case, we know $\beta^* = 1$ by Lemma 4. Then, according to Proposition 1, the β^*-lenient action is automatically self-enforcing. We do not even need either x_t or z_t to enforce this action. This completes our proof for the case when $\alpha_0 \leq \alpha < \alpha_1(\sigma)$.

Then, consider the case when $\alpha < \alpha_0(\sigma)$ and $\sigma > \sigma_0$. In this case, we know $\beta_* < \beta^* < 1$ by Lemma 4. Further, by the same lemma and their definitions, we know that $u_{l\beta^*} \geq 0$, $v_{l\beta^*} > v_{l\beta_*} > v_0$, and $\pi_{l\beta^*} \geq \pi_0$. Therefore, for the β^*-lenient action, the participation constraints of all the players are satisfied. Let us enforce the β^*-lenient action through coordination on z_t. There are three incentive constraints:

$$(1-\delta)[-c + \Pr(s=1)b + \Pr(\omega = D, s = 0)\beta^* b] + \delta u_{l\beta^*} \geq 0, \tag{$IC_{\mathcal{A}}^{\beta^*}$}$$

$$(1-\delta)\Pr(V_{\mathcal{P},1} = Y \mid r = 1, \beta^*)Y + \delta \pi_{l\beta^*} \geq \pi_0, \text{ and} \tag{$IC_{\mathcal{P}}^{\beta^*}$}$$

$$\begin{aligned}(1-\delta)v_0 + \delta v_{l\beta^*} &\geq (1-\delta)y + \delta\left[\Pr(V_{\mathcal{P}} = Y \mid s = 0)v_{l\beta^*} + \Pr(V_{\mathcal{P}} = 0 \mid s = 0)v_0\right] \\ &\geq (1-\delta)y + \delta\left[\frac{1-\sigma}{2-\sigma}v_{l\beta^*} + \frac{1}{2-\sigma}v_0\right].\end{aligned} \tag{$IC_{\mathcal{M}}^{\beta^*}$}$$

($IC_{\mathcal{A}}^{\beta^*}$) indicates that the agent is willing to exert effort. ($IC_{\mathcal{P}}^{\beta^*}$) indicates that the principal is willing to follow the manager's recommendation (note that the principal cannot observe z). Finally, ($IC_{\mathcal{M}}^{\beta^*}$) indicates that upon $z > \beta_*$, the manager is willing to send $r = 0$ when $(\omega, s) = (D, 0)$.

Observe that ($IC_{\mathcal{A}}^{\beta^*}$) is equivalent to $u_{l\beta^*} \geq 0$. Hence, it is satisfied. From the decomposition

$$\pi_{l\beta^*} = \Pr(r=1)\Pr(V_{\mathcal{P}} = Y \mid r = 1, \beta^*)Y + \Pr(r = 0)\pi_0$$

and $\pi_{l\beta^*} \geq \pi_0$ we know that $\Pr(V_{\mathcal{P}} = Y \mid r = 1, \beta^*)Y \geq \pi_0$. Then, as $\pi_{l\beta^*} \geq \pi_0$, ($IC_{\mathcal{P}}^{\beta^*}$) is satisfied. For the manager's incentive constraint, observe that the difference between the left-hand side and the right-hand side is continuous and increasing in δ, which is strictly positive when $\delta = 1$ (recall that $v_{l\beta^*} > v_0$) and strictly negative when $\delta = 0$. Hence, there exists $\underline{\delta}_1 \in (0, 1)$ such that ($IC_{\mathcal{M}}^{\beta^*}$) holds if and only if $\delta \geq \underline{\delta}_1$. Since $v_{l\beta^*} > v_{l\beta_*}$, comparing ($IC_{\mathcal{M}}^{\beta^*}$) with ($IC_{\mathcal{M}}^{\beta_*}$) shows $\underline{\delta}_1 < \bar{\delta}$. This completes our proof for the case when $\alpha < \alpha_0(\sigma)$. □

The next lemma shows that if $\sigma \in (\sigma_0, \sigma_1)$, $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ may be sustained as an equilibrium payoff by non-stationary contracts, even when the players are not sufficiently patient ($\delta < \bar{\delta}$).

Lemma 8. *Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma_0 < \sigma < \sigma_1$. There exists $\underline{\delta} \in [\underline{\delta}_1, \bar{\delta})$ (which depends on α and σ) such that if $\delta \in [\underline{\delta}, \bar{\delta})$, $\pi_{l\beta_*}$ can be sustained by the following relational contract:*

(i) *It starts with a β-lenient action in the first period for some fixed $\beta \in [0, \beta_*)$.*

(ii) *It randomizes between the β-lenient action and the β^*-lenient action in the subsequent periods.*

(iii) *Once the β^*-lenient action is realized, the future actions are always β^*-lenient.*

Proof. We intend to find $\underline{\delta} \in [\underline{\delta}_1, \bar{\delta})$ such that if $\delta \in [\underline{\delta}, \bar{\delta})$, the equilibrium can be constructed recursively as follows. Use the profile of continuation payoffs as the state variable. Initially, the state is $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$, as this is the equilibrium payoff. In this state, the players enforce some β-lenient action, where $\beta \in [0, \beta_*)$. The equilibrium payoffs then decompose into the flow payoffs $(u_{l\beta}, v_{l\beta}, \pi_{l\beta})$ from the β-lenient action and the continuation payoffs (u_c, v_c, π_c):

$$\begin{cases} u_{l\beta_*} = (1-\delta)u_{l\beta} + \delta u_c \\ v_{l\beta_*} = (1-\delta)v_{l\beta} + \delta v_c \\ \pi_{l\beta_*} = (1-\delta)\pi_{l\beta} + \delta \pi_c, \end{cases}$$

where (u_c, v_c, π_c) corresponds to a β'-lenient action for some fixed $\beta' \in (\beta_*, \beta^*)$. In the next period, the state randomizes between $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$: it equals $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ with probability $\frac{\beta^*-\beta'}{\beta^*-\beta_*}$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ with probability $\frac{\beta'-\beta_*}{\beta^*-\beta_*}$. When $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ is realized, the players repeat their behavior in the initial state. When $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ is realized, the players choose the β^*-lenient action, and the state stays there forever. Figure 7 illustrates our construction.

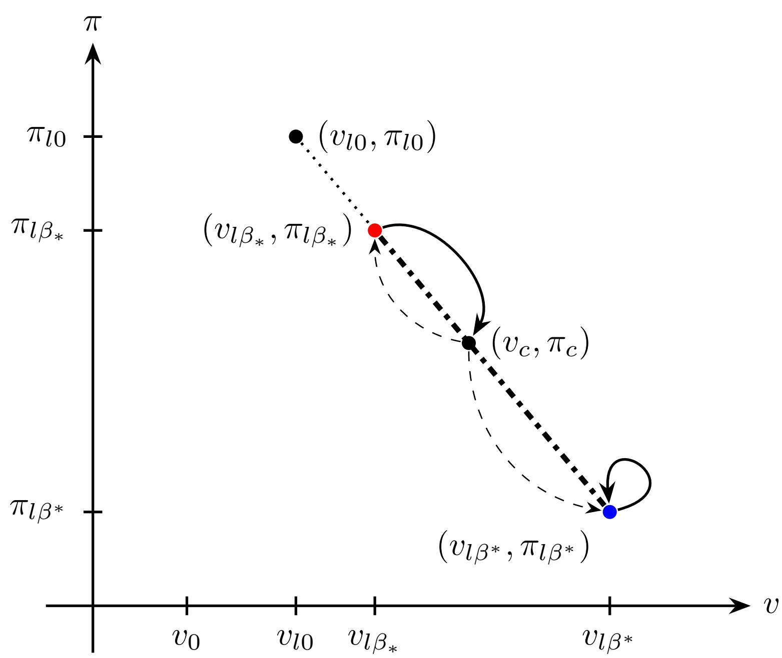

Figure 7: A non-stationary contract sustaining $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ when $\alpha < \alpha_1(\sigma)$ and $\sigma < \sigma_1$

Suppose $\sigma > \sigma_0$ and $\alpha < \alpha_1(\sigma)$. According to Lemma 7, the β^*-lenient action is self-enforcing if $\delta \geq \underline{\delta}_1$. Therefore, our construction reduces to the analysis of the initial state. By Lemma 4, we have $u_{l\beta_*} \geq 0$ and $v_{l\beta_*} > v_0$ for $\sigma > \sigma_0$. Further, as $\beta_* < \beta^*$ for $\sigma > \sigma_0$, $\pi_{l\beta_*} > \pi_{l\beta^*} \geq \pi_0$. Hence, all the participation constraints are satisfied in the initial state. It remains to show the β-lenient action and how to enforce this action.

First, in addition to $\sigma > \sigma_0$ and $\alpha < \alpha_1(\sigma)$, suppose $\alpha \geq \alpha_0(\sigma)$. In this case, let us enforce the β-lenient action through coordination on x_1. We intend to show that there exist $\beta \in [0, \beta_*)$ and $\underline{\delta} \in (0, \bar{\delta})$ such that when $\delta \in [\underline{\delta}, \bar{\delta})$, (u_c, v_c, π_c) is between $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$, and the following incentive constraints are satisfied:

$$(1-\delta)[-c + \Pr(s_1 = 1)b + \Pr(\omega_1 = D, s_1 = 0)\beta b] + \delta u_c \geq 0, \qquad (IC_{\mathcal{A}}^{\beta})$$

$$(1-\delta)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 \leq \beta)Y + \delta \pi_c \geq \pi_0, \qquad (IC_{\mathcal{P},1}^{\beta})$$

$$(1-\delta)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 > \beta)Y + \delta \pi_c \geq \pi_0, \text{ and} \qquad (IC_{\mathcal{P},2}^{\beta})$$

$$\begin{aligned}(1-\delta)v_0 + \delta v_c &\geq (1-\delta)y + \delta\left[\Pr(V_{\mathcal{P},1} = Y \mid s_1 = 0)v_c + \Pr(V_{\mathcal{P},1} = 0 \mid s_1 = 0)v_0\right] \qquad (IC_{\mathcal{M}}^{\beta})\\ &\geq (1-\delta)y + \delta\left[\frac{1-\sigma}{2-\sigma}v_c + \frac{1}{2-\sigma}v_0\right].\end{aligned}$$

($IC_{\mathcal{A}}^{\beta}$) indicates that the agent is willing to exert effort. ($IC_{\mathcal{P},1}^{\beta}$) and ($IC_{\mathcal{P},2}^{\beta}$) indicate that the principal is willing to follow the manager's recommendation. Finally, ($IC_{\mathcal{M}}^{\beta}$) indicates that upon $x_1 > \beta_*$, the manager is willing to send $r_1 = 0$ when $(\omega_1, s_1) = (D, 0)$.

To pin down β, write (u_c, v_c, π_c) as

$$u_c = \frac{u_{l\beta_*} - (1-\delta)u_{l\beta}}{\delta}, v_c = \frac{v_{l\beta_*} - (1-\delta)v_{l\beta}}{\delta}, \text{ and } \pi_c = \frac{\pi_{l\beta_*} - (1-\delta)\pi_{l\beta}}{\delta}.$$

Notice that $u_{l\beta}$ and $v_{l\beta}$ are strictly increasing in β, and $\pi_{l\beta}$ is strictly decreasing β. Hence, as β converges to β_* from below, u_c converges to $u_{l\beta_*}$ from above, v_c converges to $v_{l\beta_*}$ from above, and π_c converges to $\pi_{l\beta_*}$ from below. Then, given $\delta = \bar{\delta}$, there exists $\beta^* < \beta < \beta_*$ such that $u_c \in (u_{l\beta_*}, u_{l\beta^*})$, $v_c \in (v_{l\beta_*}, v_{l\beta^*})$, and $\pi_c \in (\pi_{l\beta^*}, \pi_{l\beta_*})$. Fix any such β. As (u_c, v_c, π_c) is continuous in δ, there exists $\underline{\delta}_2 < \bar{\delta}$ such that when $\delta \in [\underline{\delta}_2, \bar{\delta})$, $u_c \in (u_{l\beta_*}, u_{l\beta^*})$, $v_c \in (v_{l\beta_*}, v_{l\beta^*})$, and $\pi_c \in (\pi_{l\beta^*}, \pi_{l\beta_*})$. That is, for the fixed β, (u_c, v_c, π_c) is indeed between $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ if $\delta \in [\underline{\delta}_2, \bar{\delta})$.

Then, we look into the incentive constraints. Observe that ($IC_{\mathcal{A}}^{\beta}$) is equivalent to

$$(1-\delta)u_{l\beta} + \delta u_c \geq 0.$$

Since $(1-\delta)u_{l\beta} + \delta u_c = u_{l\beta_*} \geq 0$, ($IC_{\mathcal{A}}^{\beta}$) is satisfied. For the principal's constraints, from the decomposition

$$\pi_{l\beta} = \Pr(r_1 = 1, x_1 \leq \beta)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 \leq \beta)Y + \Pr(r_1 = 0, x_1 \leq \beta)\pi_0$$

and $\pi_{l\beta} > \pi_{l\beta_*} > \pi_{l\beta^*} \geq \pi_0$ we know that $\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 \leq \beta)Y \geq \pi_0$. Then, because $\pi_c > \pi_{l\beta^*} \geq \pi_0$, ($IC^{\beta}_{\mathcal{P},1}$) is satisfied. And since $\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 \leq \beta)Y \leq \Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 > \beta)Y$, ($IC^{\beta}_{\mathcal{P},2}$) is also satisfied. For the manager's incentive constraint, observe that the difference between the left-hand side and the right-hand side is continuous and increasing in δ, which is strictly positive when $\delta = 1$ ($v_c > v_{l\beta_*} > v_0$) and strictly negative when $\delta = 0$. Hence, there exists $\underline{\delta}_3 \in (0,1)$ such that ($IC^{\beta}_{\mathcal{M}}$) holds if and only if $\delta \geq \underline{\delta}_3$. As $v_c > v_{l\beta_*}$ and ($IC^{\beta}_{\mathcal{M}}$) is easier to satisfy when v_c is larger, comparing this incentive constraint with ($IC^{\beta_*}_{\mathcal{M}}$) shows $\underline{\delta}_3 < \bar{\delta}$.

Now, instead of assuming $\alpha \geq \alpha_0(\sigma)$, consider the case when $\alpha < \alpha_0(\sigma)$. In this case, we follow the same construction as above to define β, $\underline{\delta}_2$, and $\underline{\delta}_3$. However, since $\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, x_1 \leq \beta_*)Y < \pi_0$, ($IC^{\beta_*}_{\mathcal{P},1}$) may be violated.

We can resolve this issue with the private randomizing device z_t. Instead of x_t, let us enforce the β-lenient action through coordination on z_t. Then, both ($IC^{\beta_*}_{\mathcal{A}}$) and ($IC^{\beta_*}_{\mathcal{M}}$) remain the same. However, as the principal cannot observe z_1, ($IC^{\beta_*}_{\mathcal{P},1}$) becomes

$$(1-\delta)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, \beta_*)Y + \delta\pi_c \geq \pi_0.$$

Since $\pi_{l\beta_*} \geq \pi_0$ and $\pi_{l\beta_*} = \Pr(r_1 = 1)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, \beta_*)Y + \Pr(r_1 = 0)\pi_0$, we have $\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1, \beta_*)Y \geq \pi_0$. Together with $\pi_c \geq \pi_0$, this implies that the above incentive constraint is satisfied. With this fact established, the argument in the previous construction follows.

Finally, let $\underline{\delta} := \max\{\underline{\delta}_1, \underline{\delta}_2, \underline{\delta}_3\}$. Note that $\underline{\delta} < \bar{\delta}$. For arbitrarily chosen $\delta \in [\underline{\delta}, \bar{\delta})$, denote as β' the lenience level that induces $v_c = v_{l\beta'}$. As

$$\beta' = \frac{\beta^* - \beta'}{\beta^* - \beta_*}\beta_* + \frac{\beta' - \beta_*}{\beta^* - \beta_*}\beta^*,$$

we can verify that the proposed randomization between $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ indeed generates (u_c, v_c, π_c). This completes our construction. □

Now, suppose $\alpha < \alpha_1(\sigma)$ but $\sigma \geq \sigma_1$. In this case, if $\delta < \bar{\delta}$, $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ cannot be sustained as an equilibrium payoff. This is because when $\sigma \geq \sigma_1$, $\beta_* = 0$ (Lemma 4), and $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ represents an extreme point in the set of payoff profiles achievable from some action. If we still aim to use $(u_{l\beta_*}, v_{l\beta_*}, \pi_{l\beta_*})$ as the initial equilibrium state, we are unable to identify a supporting action. For this scenario, the following lemma introduces an alternative upper bound on the principal's equilibrium payoff.

Lemma 9. *Suppose $\sigma \geq \sigma_1$ and $\delta < \bar{\delta}$. There exist $\hat{\delta} \in (0, \bar{\delta})$ and $\bar{\pi}(\delta)$ (both of which depend on α and σ) such that if $\delta \in [\hat{\delta}, \bar{\delta})$, then in any equilibrium, the principal's payoff is weakly smaller than $\bar{\pi}(\delta)$. Moreover, for $\delta \in [\hat{\delta}, \bar{\delta})$, $\bar{\pi}(\delta)$ cannot be sustained as an equilibrium payoff by any stationary contract.*

Proof. We begin the proof by finding the the smallest continuation payoff that incentivizes the manager to recommend $r = 0$ when $(\omega, s) = (D, 0)$. Consider the manager's incentive constraint, ($IC^{\beta}_{\mathcal{M}}$). Taking other variables and parameters as fixed, we solve the value of v_c such that the constraint is binding. The result is

$$\tilde{v}(\delta) := \frac{(1-\delta)(2-\sigma)(y-v_0)}{\delta} + v_0.$$

According to Lemma 6 and its proof, $\tilde{v}(\bar{\delta}) = v_{l\beta_*}$, and if $\delta < \bar{\delta}$, then $\tilde{v}(\delta) > v_{l\beta_*}$.

Observe that $\tilde{v}(\delta)$ is strictly decreasing and continuous in δ. When $\sigma \geq \sigma_1$, $\beta^* > \beta_* = 0$ (Lemma 4), and thus $v_{l\beta_*} < v_{l\beta^*}$. Let $\hat{\delta} \in (0, \bar{\delta})$ such that $\tilde{v}(\hat{\delta}) = v_{l\beta^*}$. Then, if $\delta \in [\hat{\delta}, \bar{\delta})$, we have $v_{l\beta_*} < \tilde{v}(\delta) \leq v_{l\beta^*}$, and there exists $\tilde{\beta}(\delta) \in (\beta_*, \beta^*]$ such that $v_{l\tilde{\beta}(\delta)} = \tilde{v}(\delta)$. Denote the payoff profile from the $\tilde{\beta}(\delta)$-lenient action as $(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta))$. Since the $\tilde{\beta}(\delta)$-lenient action implements the project whenever $s = 1$, it follows that $\tilde{\pi}(\delta)$ is the largest payoff that the principal can obtain, subject to the constraint that the manager's payoff is at least $\tilde{v}(\delta)$.

In the following, suppose $\delta \in [\hat{\delta}, \bar{\delta})$. Define $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ as

$$\begin{cases} \bar{u}(\delta) := (1-\delta)u_{l0} + \delta\tilde{u}(\delta) \\ \bar{v}(\delta) := (1-\delta)v_{l0} + \delta\tilde{v}(\delta) \\ \bar{\pi}(\delta) := (1-\delta)\pi_{l0} + \delta\tilde{\pi}(\delta). \end{cases} \tag{17}$$

That is, $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ is achieved by taking the 0-lenient action in the current period and promising $(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta))$ as the continuation payoffs. Since $\beta_* < \tilde{\beta}(\delta) \leq \beta^* \leq 1$ and $\beta_* = 0$, $\pi_{l0} > \tilde{\pi}(\delta) \geq \pi_{l\beta^*} \geq \max\{\pi_{l1}, \pi_0\}$. Then, from (17) we know $\bar{\pi}(\delta) > \max\{\pi_{l1}, \pi_0\}$.

To prove the lemma, we show the following result: when $\delta \in [\hat{\delta}, \bar{\delta})$, there exists no equilibrium in which the principal's continuation payoff at the beginning of some period is strictly greater than $\bar{\pi}(\delta)$ conditional on the manager's history.

Suppose not. Let $\boldsymbol{\sigma}_{\mathsf{eq}}$ denote the strategy profile for a given equilibrium eq. Consider

$$\Pi^{\mathcal{M}} := \{\mathbb{E}[\pi_\tau(h^{\mathcal{P},\tau}, \boldsymbol{\sigma}_{\mathsf{eq}}) \mid h^{\mathcal{M},\tau}] : \tau \geq 1, \mathsf{eq} \text{ is an equilibrium, for which } h^{\mathcal{M},\tau} \text{is an on-path history of the manager at period } \tau\},$$

which is the set of the principal's continuation payoffs that can arise conditional on the manager's history at some period in some equilibrium. Since this set is bounded above by Y, it admits a finite supremum. Denote this supremum as ψ. By our assumption, we know that $\psi > \bar{\pi}(\delta)$.

Next, we intend to find $\varepsilon > 0$ that has the following properties: (i) $\psi - \varepsilon > \bar{\pi}(\delta)$, and (ii) if $\mathbb{E}[\pi_\tau(h^{\mathcal{P},\tau}, \boldsymbol{\sigma}_{\mathsf{eq}}) \mid h^{\mathcal{M},\tau}] > \psi - \varepsilon$, the action at the period τ is neither the 1-lenient action nor the outside option.

Since $\psi > \bar{\pi}(\delta) > \max\{\pi_{l1}, \pi_0\}$, we can pick $\varepsilon > 0$ such that

$$\psi - \varepsilon > \bar{\pi}(\delta), \ \frac{\psi - \varepsilon - (1-\delta)\pi_{l1}}{\delta} > \psi, \text{ and } \frac{\psi - \varepsilon - (1-\delta)\pi_0}{\delta} > \psi.$$

Now suppose that for this pick of ε, $\mathbb{E}[\pi_\tau(h^{\mathcal{P},\tau}, \boldsymbol{\sigma}_{\mathsf{eq}}) \mid h^{\mathcal{M},\tau}] > \psi - \varepsilon$. Let $\pi_{\mathsf{current},\tau} \in \{\pi_{l1}, \pi_0\}$ in the following decomposition:

$$\mathbb{E}[\pi_\tau(h^{\mathcal{P},\tau}, \boldsymbol{\sigma}_{\mathsf{eq}}) \mid h^{\mathcal{M},\tau}] = \mathbb{E}\left[(1-\delta)\pi_{\text{current},\tau} + \delta\pi_{\text{continuation},\tau} \mid h^{\mathcal{M},\tau}\right], \tag{18}$$

where $\pi_{\text{current},\tau}$ denotes the flow payoff from the action at period τ, and $\pi_{\text{continuation},\tau}$ the principal's continuation payoff at the beginning of period $\tau+1$. Now, from the second and third condition on ε stated above, we get $\mathbb{E}[\pi_{\text{continuation},\tau} \mid h^{\mathcal{M},\tau}] > \psi$. However, then, this would imply that for some on-path history $h^{\mathcal{M},\tau+1}$, $\mathbb{E}[\pi_{\tau+1}(h^{\mathcal{P},\tau+1}, \boldsymbol{\sigma}_{\mathsf{eq}}) \mid h^{\mathcal{M},\tau+1}] > \psi$, which contradicts with $\psi = \sup \Pi^{\mathcal{M}}$. Therefore, this ε is indeed what we want.

Now, we can show that $\psi > \bar{\pi}(\delta)$ leads to a contraction. Choose an equilibrium eq^* in which there exist a period τ^* and an on-path history $h^{\mathcal{M},\tau^*}$ such that

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}] > \psi - \varepsilon.$$

According to the result in the previous paragraph, the action in period τ^* cannot be the outside option. Hence, the agent must exert effort in this particular period. For eq^* and $h^{\mathcal{M},\tau^*}$, define

$$\mathsf{XZ}_0 := \{(x_{\tau^*}, z_{\tau^*}) \in [0,1]^2 : \text{ given } (x_{\tau^*}, z_{\tau^*}),\, r_{\tau^*} = 0 \text{ if } (\omega_{\tau^*}, s_{\tau^*}) = (D, 0)\}, \text{ and}$$

$$\mathsf{XZ}_1 := \{(x_{\tau^*}, z_{\tau^*}) \in [0,1]^2 : \text{ given } (x_{\tau^*}, z_{\tau^*}),\, r_{\tau^*} = 1 \text{ whenever } (\omega_{\tau^*}, s_{\tau^*}) = (D, 0)\}.$$

Clearly, $\mathsf{XZ}_0 \cap \mathsf{XZ}_1 = \emptyset$ and $\mathsf{XZ}_0 \cup \mathsf{XZ}_1 = [0,1]^2$. In the following, we show:

(i) $\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_0] \leq \bar{\pi}(\delta)$.

(ii) $\mathbb{E}[\pi_{\tau^*+1}(h^{\mathcal{P},\tau^*+1}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1] > \psi$, implying that $\Pr(\mathsf{XZ}_1) = 0$.

First, suppose $(x_{\tau^*}, z_{\tau^*}) \in \mathsf{XZ}_0$. Consider a similar decomposition to (18):

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})] = \mathbb{E}\left[(1-\delta)\pi_{\text{current},\tau^*} + \delta\pi_{\text{continuation},\tau^*} \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})\right],$$

where the main difference is that now the expectation is also conditional on the realization of the randomizing devices. As the 0-lenient action maximizes the principal's current payoff, $\pi_{\text{current},\tau^*} \leq \pi_{l0}$. Furthermore, we want to show $\mathbb{E}[\pi_{\text{continuation},\tau} \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})] \leq \tilde{\pi}(\delta)$.

Observe that in this case, it must be incentive compatible for the manager to send $r_{\tau^*} = 0$ when $(\omega_{\tau^*}, s_{\tau^*}) = (D, 0)$, and recall that $\tilde{v}(\delta)$ is the smallest continuation payoff to satisfy the manager's incentive constraint ($IC^{\beta}_{\mathcal{M}}$). It then follows that $\mathbb{E}[v_{\text{continuation},\tau^*} \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})] \geq \tilde{v}(\delta)$. Hence, $\mathbb{E}[\pi_{\text{continuation},\tau^*} \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})] \leq \tilde{\pi}(\delta)$. Taking into account these facts, we conclude that

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})] \leq \mathbb{E}\left[(1-\delta)\pi_{l0} + \delta\tilde{\pi}(\delta) \mid h^{\mathcal{M},\tau^*}, (x_{\tau^*}, z_{\tau^*})\right] = \bar{\pi}(\delta).$$

As indicated by part (i), $\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_0] \leq \bar{\pi}(\delta)$.

Second, let us look into $\mathbb{E}[\pi_{\tau^*+1}(h^{\mathcal{P},\tau^*+1}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1]$, the principal's payoff at the beginning of period $\tau^* + 1$ conditional on $h^{\mathcal{M},\tau^*}$ and XZ_1. Note the following decomposition:

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}] = \Pr(\mathsf{XZ}_0)\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_0] + \Pr(\mathsf{XZ}_1)\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1]. \tag{19}$$

Since

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}] > \psi - \varepsilon > \bar{\pi}(\delta) \geq \mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_0],$$

$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1] > \psi - \varepsilon$. Then, observe the following decomposition:

$$\mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1] = \mathbb{E}\left[(1-\delta)\pi_{\text{current},\tau^*} + \delta\pi_{\text{continuation},\tau^*} \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1\right].$$

Given XZ_1, the manager sends $r_{\tau^*} = 1$ as long as $(\omega_{\tau^*}, s_{\tau^*}) = (D, 0)$. Hence, $\pi_{\text{current},\tau^*} \leq \pi_{l1}$. From the way we pick ε, it follows that

$$\mathbb{E}[\pi_{\tau^*+1}(h^{\mathcal{P},\tau^*+1}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1] = \mathbb{E}[\pi_{\text{continuation},\tau^*} \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_1] > \psi.$$

However, this would imply that $\mathbb{E}[\pi_{\tau^*+1}(h^{\mathcal{P},\tau^*+1}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*+1}] > \psi$ for some on-path history $h^{\mathcal{M},\tau^*+1}$, which contradicts with $\psi = \sup \Pi^{\mathcal{M}}$. Consequently, $\Pr(\mathsf{XZ}_1) = 0$, as indicated by part (ii).

Substituting what we have known from parts (i) and (ii) into (19), we have

$$\psi - \varepsilon < \mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}] = \mathbb{E}[\pi_{\tau^*}(h^{\mathcal{P},\tau^*}, \boldsymbol{\sigma}_{\mathsf{eq}^*}) \mid h^{\mathcal{M},\tau^*}, \mathsf{XZ}_0] \leq \bar{\pi}(\delta),$$

which contradicts with $\psi - \varepsilon > \bar{\pi}(\delta)$. This completes the proof of our claim, i.e., $\psi \leq \bar{\pi}(\delta)$. In particular, the principal's equilibrium payoff is bounded above by $\bar{\pi}(\delta)$.

Finally, we show that $\delta \in [\hat{\delta}, \bar{\delta})$, $\bar{\pi}(\delta)$ cannot be sustained as an equilibrium payoff by any stationary contract.

Denote as $\bar{\beta}(\delta)$ the lenience that induces the payoff profile $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$. Since $\bar{v}(\delta) < \tilde{v}(\delta) = v_{l\tilde{\beta}(\delta)}$ and $\tilde{\beta}(\delta) \leq 1$, we know that for the $\bar{\beta}(\delta)$-lenient action to be self-enforcing, it must be incentive compatible for the manager to send $r = 0$ when $(\omega, s) = (D, 0)$. However, recall that $\tilde{v}(\delta)$ is the smallest continuation payoff to satisfy the manager's incentive constraint ($IC_{\mathcal{M}}^{\beta}$). As $\bar{v}(\delta) < \tilde{v}(\delta)$, the $\bar{\beta}(\delta)$-lenient action cannot be self-enforcing. As a result, $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ is not an equilibrium payoff from any stationary contract. □

Similar to Lemma 5, Lemma 9 implies that when $\sigma \geq \sigma_1$ and $\delta \in [\hat{\delta}, \bar{\delta})$, a relational contract is automatically optimal if it gives the principal a payoff of $\bar{\pi}(\delta)$. The following lemma shows one such contract.

Lemma 10. *Suppose $\alpha < \alpha_1(\sigma)$, $\sigma \geq \sigma_1$, and $\max\{\underline{\delta}_1, \hat{\delta}\} \leq \delta < \bar{\delta}$. Then, $\bar{\pi}(\delta)$ can be sustained by the following relational contract:*

(*i*) *It starts with the 0-lenient action in the first period.*

(*ii*) *It randomizes between the* 0*-lenient action and the* β^**-lenient action in the subsequent periods.*

(*iii*) *Once the* β^**-lenient action is realized, the future actions are always* β^**-lenient.*

Proof. Similar to the proof of Lemma 8, we intend to construct the equilibrium recursively by using continuation payoffs as the state variable. Initially, the state is $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$, as this is the equilibrium payoff. In this state, the players choose the 0-lenient action, and the payoffs decompose into the flow payoffs from the 0-lenient action and the continuation payoffs $(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta))$; see (17). Recall from the proof of Lemma 9 that $\bar{v}(\delta) < \tilde{v}(\delta) \leq v_{l\beta^*}$, and $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ and $(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta))$ are both convex combinations of $(u_{l0}, v_{l0}, \pi_{l0})$ and $(u_{l1}, v_{l1}, \pi_{l1})$. Hence, $(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta))$ can be written as a convex combination of $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$:

$$(\tilde{u}(\delta), \tilde{v}(\delta), \tilde{\pi}(\delta)) = \frac{v_{l\beta^*} - \tilde{v}(\delta)}{v_{l\beta^*} - \bar{v}(\delta)}(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta)) + \frac{\tilde{v}(\delta) - \bar{v}(\delta)}{v_{l\beta^*} - \bar{v}(\delta)}(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*}).$$

Then, in the next period, let the state randomize between $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$. It equals $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ with probability $\frac{v_{l\beta^*} - \tilde{v}(\delta)}{v_{l\beta^*} - \bar{v}(\delta)}$ and $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ with probability $\frac{\tilde{v}(\delta) - \bar{v}(\delta)}{v_{l\beta^*} - \bar{v}(\delta)}$. When $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ is realized, the players repeat their behavior in the initial state. When $(u_{l\beta^*}, v_{l\beta^*}, \pi_{l\beta^*})$ is realized, the players choose the β^*-lenient action, and the state stays there forever.

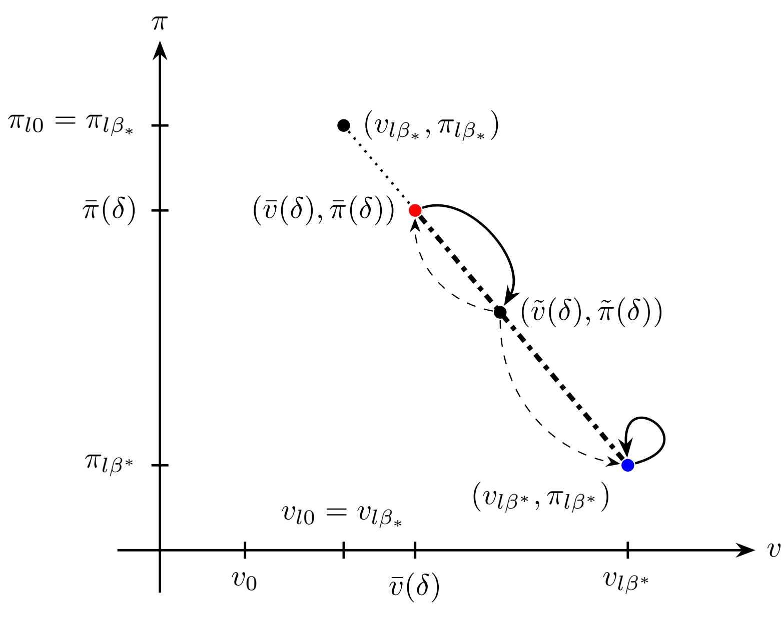

Figure 8: A non-stationary contract sustaining $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ when $\alpha < \alpha_1(\sigma)$ and $\sigma \geq \sigma_1$

Suppose $\alpha < \alpha_1(\sigma)$, $\sigma \geq \sigma_1$, and $\max\{\underline{\delta}_1, \hat{\delta}\} \leq \delta < \bar{\delta}$. According to Lemma 7, if $\delta \geq \underline{\delta}_1$, the β^*-lenient action is self-enforcing. Therefore, our construction reduces to the analysis of the initial state. By Lemma 4 and the definition of $(\bar{u}(\delta), \bar{v}(\delta), \bar{\pi}(\delta))$ (see the proof of Lemma 9), we have

$\bar{u}(\delta) \geq u_{l0} \geq 0$ and $\bar{v}(\delta) > v_0$, and $\bar{\pi}(\delta) > \tilde{\pi}(\delta) \geq \pi_{l\beta^*} \geq \pi_0$. Hence, all the participation constraints are satisfied in the initial state. It remains to show how to enforce the 0-lenient action.

Notice that this 0-lenient action can be enforced without the randomization devices. Simply, the agent exerts effort, and the manager sends $r_1 = 1$ if and only if $s_1 = 1$. The principal follows the manager's recommendation. When any player is publicly known to have deviated, the repeated game is terminated. In this case, there are three incentive constraints:

$$(1-\delta)[-c + \Pr(s_1 = 1)b] + \delta\tilde{u}(\delta) \geq 0, \qquad (IC^0_{\mathcal{A}})$$

$$(1-\delta)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1)Y + \delta\tilde{\pi}(\delta) \geq \pi_0, \text{ and} \qquad (IC^0_{\mathcal{P}})$$

$$\begin{aligned}(1-\delta)v_0 + \delta\tilde{v}(\delta) &\geq (1-\delta)y + \delta\left[\Pr(V_{\mathcal{P},1} = Y \mid s_1 = 0)\tilde{v}(\delta) + \Pr(V_{\mathcal{P},1} = 0 \mid s_1 = 0)v_0\right] \\ &\geq (1-\delta)y + \delta\left[\frac{1-\sigma}{2-\sigma}\tilde{v}(\delta) + \frac{1}{2-\sigma}v_0\right]. \qquad (IC^0_{\mathcal{M}})\end{aligned}$$

$(IC^0_{\mathcal{A}})$ indicates that the agent is willing to exert effort. $(IC^0_{\mathcal{P}})$ indicates that the principal is willing to follow the manager's recommendation. Finally, $(IC^0_{\mathcal{M}})$ indicates that the manager is willing to send $r_1 = 0$ when $(\omega_1, s_1) = (D, 0)$.

Then, we look into the incentive constraints. Observe that $(IC^0_{\mathcal{A}})$ is equivalent to

$$(1-\delta)u_{l0} + \delta\tilde{u}(\delta) \geq 0.$$

Since $\tilde{u}(\delta) \geq u_{l0} \geq 0$, $(IC^0_{\mathcal{A}})$ is satisfied. For the principal's constraints, from the decomposition

$$\pi_{l0} = \Pr(r_1 = 1)\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1)Y + \Pr(r_1 = 0)\pi_0$$

and $\pi_{l0} \geq \pi_0$ we know that $\Pr(V_{\mathcal{P},1} = Y \mid r_1 = 1)Y \geq \pi_0$. Then, because $\tilde{\pi}(\delta) \geq \pi_{l\beta^*} \geq \pi_0$, $(IC^0_{\mathcal{P}})$ is satisfied. For the manager's incentive constraint, observe that it is binding as $\tilde{v}(\delta)$ binds $(IC^{\beta}_{\mathcal{M}})$. This completes our construction. □

Now we can prove Proposition 4.

Proof of Proposition 4. Suppose $\alpha < \alpha_1(\sigma)$ and $\sigma \geq \sigma_0$. Part (i) is immediate from Lemma 5 and Lemma 6. For part (ii), we first consider the case when $\sigma \in (\sigma_0, \sigma_1)$. The result is immediate from Lemma 5, Lemma 6, and Lemma 8. Then, consider the case when $\sigma \in [\sigma_1, 1]$. In this case, the result follows from Lemma 9 and Lemma 10. □